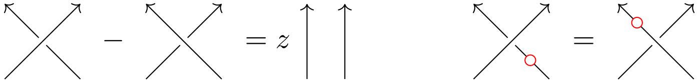
Quantum Heisenberg categorification



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Joint work with J. Brundan and B. Webster

Outline

Goal:

1. Define a family of quantum Heisenberg categories categorifying the Heisenberg algebra
2. Study categorical actions and applications in representation theory

Overview:

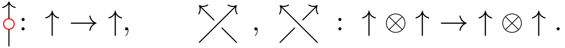
1. Quantum Heisenberg category
2. Categorical actions
3. Quantum Frobenius Heisenberg category
4. Future directions

Monoidally generated affine Hecke algebras

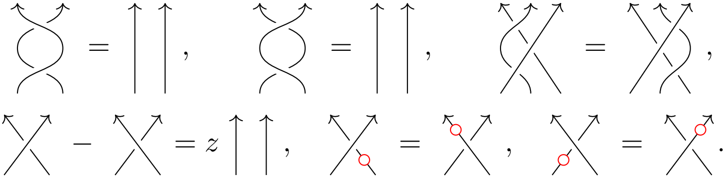
Fix a commutative ground ring $\Bbbk$ and parameters $z, t \in \Bbbk^\times$.

Let $\mathcal{AH}(z)$ be the strict $\Bbbk$-linear monoidal category generated by

- one object $\uparrow$, and
- three morphisms



subject to the relations



Then

$$\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$$

is the affine Hecke algebra of type A_{n-1}.

The quantum Heisenberg category

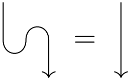
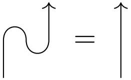
Fix a central charge $k \in \mathbb{Z}$. (Assume $k < 0$ for simplicity.)

To obtain the quantum Heisenberg category $\mathcal{H}eis_k(z,t)$ from $\mathcal{AH}(z)$ we perform two steps:

1. We adjoin a right dual $\downarrow$ to $\uparrow$. Precisely, we add a generating object $\downarrow$ and additional generating morphisms

    $: \mathbb{1} \to \downarrow \otimes \uparrow$ and  $: \uparrow \otimes \downarrow \to \mathbb{1}$

   such that

    and  .

2. We add more generating morphisms and relations ensuring that the resulting monoidal category is pivotal and that

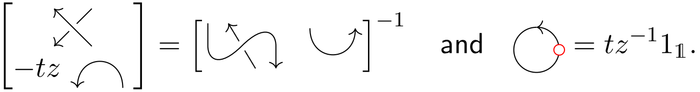
$$\downarrow \otimes \uparrow \;\cong\; \uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)}. \qquad \text{(Canonical commutation isom)}$$

The quantum Heisenberg category

There are three equivalent ways to do this. For simplicity, suppose $k = -1$.

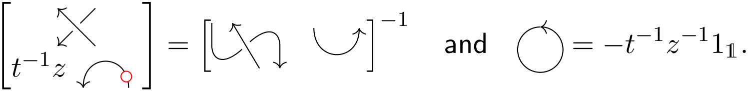
First approach: Add generating morphisms

and relations



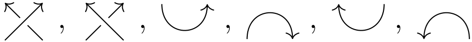
Second approach: Add generating morphisms

and relations

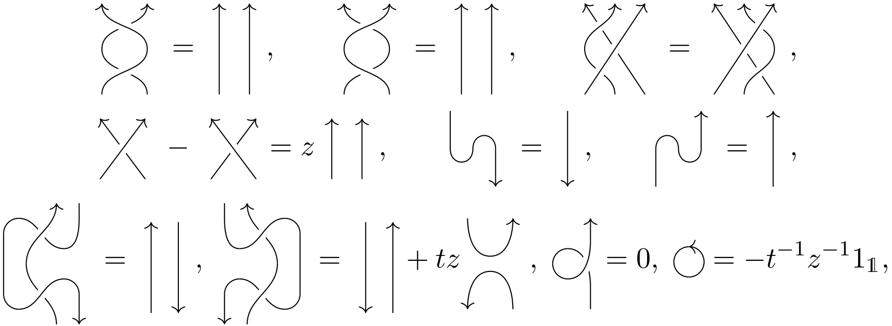


The quantum Heiseberg category

Third approach: Generating morphisms

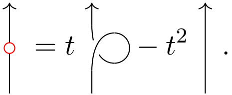


subject to the relations



and one more relation (†).

Third approach: Note that we do not need the dot generator. It can be recovered via



The extra relation (†) is that this dot is invertible.

Theorem (Brundan–S.–Webster)

1. All three approaches define isomorphic categories ($\mathcal{H}eis_k(z,t)$).
2. $\mathcal{H}eis_k(z,t)$ is strictly pivotal (i.e. we have isotopy invariance for morphisms).

Special cases

Deformed Heisenberg category ($k = -1$)

$\mathcal{H}eis_{-1}(z,t)$ is closely related to a deformed Heisenberg category $\mathcal{H}(q^2)$ introduced by Licata–S. (2013).

Precisely, $\mathcal{H}(q^2)$ is the monoidal subcategory of

$$\mathcal{H}eis_{-1}(z, -z^{-1}), \quad z = q - q^{-1},$$

consisting of all objects and morphisms that do not involve negative powers of the dots.

Affine oriented skein category ($k = 0$)

$\mathcal{H}eis_0(z,t)$ is the affine oriented skein category, an affinization of the HOMFLY-PT skein category.

Categorical actions ($k \neq 0$)

When $k \neq 0$, the category $\mathcal{H}eis_k(z,t)$ acts naturally on modules for cyclotomic Hecke algebras H_n^f of level $|k|$.

We have a chain of algebras

$$\Bbbk = H_0^f \subseteq H_1^f \subseteq H_2^f \subseteq \cdots .$$

If $k < 0$, then

- $\uparrow$ acts by induction from H_n^f-mod to H_{n+1}^f-mod,
- $\downarrow$ acts by restriction from H_n^f-mod to H_{n-1}^f-mod.

The morphisms (diagrams) act by certain natural transformations.

Fact that $\mathcal{H}eis_k(z,t)$ is pivotal corresponds to fact that induction and restriction are biadjoint.

In other words H_n^f is a Frobenius extension of H_{n-1}^f.

Categorical actions ($k = 0$)

Suppose $k = 0$ and $t = q^n$.

$\mathcal{H}eis_0(z,t)$ acts on representations of $U_q(\mathfrak{gl}_n)$:

- $\uparrow$ tensors with natural module V,
- $\downarrow$ tensors with dual V^*.

This action extends the monoidal functor

$$\text{HOMFLY-PT skein category} \to \text{cat of fd } U_q(\mathfrak{gl}_n)\text{-modules}$$

originally constructed by Turaev.

The center of $\mathcal{H}eis_0(z,t)$ maps surjectively to the center of $U_q(\mathfrak{gl}_n)$. So we get a diagrammatic calculus for this center.

Basis theorem

For many applications, one needs to know a basis for morphism spaces in $\mathcal{H}eis_k(z,t)$.

Usual approach: Use

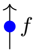
- categorical actions described above,
- known bases for the algebras involved (H_n^f and $U_q(\mathfrak{gl}_n)$),
- asymptotic faithfulness (as $n \to \infty$).

However, this approach fails for $\mathcal{H}eis_k(z,t)$, $k \neq 0$, due to $\mathcal{H}eis_k(z,t)$ having a larger center than expected.

Solution: Use "unfurling" technique of B. Webster. See upcoming talk of J. Brundan.

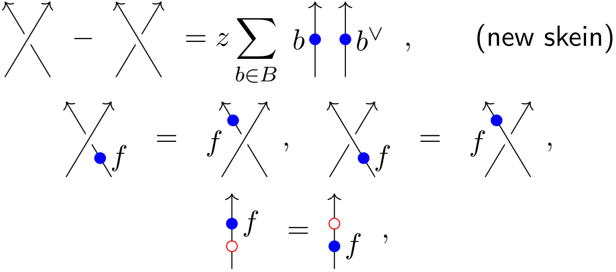
Generally, can incorporate a Frobenius superalgebra F to get a more general quantum Frobenius Heisenberg category.

Strand can now carry tokens:



, $f \in F$.

We have additional/modified relations:



$+$ inversion, etc.

Categorical actions

Categorical actions: Largely unexplored.

Case: $k = 0$

Obtain a Frobenius deformation of the affine oriented skein category.

Natural action is an open question for general F.

Should act on modules for some F-deformation of $U_q(\mathfrak{gl}_n)$.

Case: $k \neq 0$

Acts on cyclotomic quotients of affine Frobenius Hecke algebras.

The structure theory and rep theory of these algebras is work in progress (with D. Rosso).

Future directions

Traces

The trace of the deformed Heisenberg category $\mathcal{H}(q^2)$ was computed by Cautis–Lauda–Licata–Samuelson–Sussan.

It is related to the elliptic Hall algebra.

One should be able to extend this description to the larger quantum Heisenberg category $\mathcal{H}eis_k(z,t)$.

Connections to Kac–Moody 2-categories (with Brundan & Webster)

Given certain categorical Heisenberg actions, one can define a categorical Kac-Moody action.

Conversely, given certain categorical Kac–Moody actions, one can define a categorical Heisenberg action.

This extends work with Queffelec and Yacobi, which considered the level one case.