Affine Frobenius-Hecke algebras

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AMS 2018 Fall Western Sectional Meeting
San Francisco State University

October 27, 2018

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$$T_i T_j = T_j T_i \qquad |i-j| > 1$$
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Frobenius (Super)Algebras

Definition

A *Frobenius* superalgebra A is a finite dimensional associative $\Bbbk$-superalgebra with an even $\Bbbk$-linear *trace* map

$$\mathrm{tr} : A \to \Bbbk$$

such that $\ker(\mathrm{tr})$ does not contain any nonzero left ideals of A.

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We assume for simplicity that $\psi = \mathrm{Id}$ (A is symmetric).

Frobenius-Hecke algebra $H_n(A, z)$

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Generated by $A^{\otimes n}$, $T_1, \dots, T_{n-1}$, with relations

$$\begin{aligned} T_i^2 &= zt_{i,i+1} T_i + 1 && 1 \le i \le n-1 \\ T_i T_j &= T_j T_i && |i-j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && 1 \le i \le n-2 \\ T_i a_j &= a_{s_i(j)} T_i && 1 \le i \le n-1, 1 \le j \le n \end{aligned}$$

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where $t_{i,j} = \sum_{b \in B} b_i b_j^\vee$.

Special Cases

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$$A = \mathbb{k}$$
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Iwahori-Hecke
algebras (type A)

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$$H_n(A, z)$$

$A = \mathbb{k}$, $z = q - q^{-1}$ → Iwahori-Hecke algebras (type A)

$A = \mathbb{k}[C_d]$, $z = (q - q^{-1})/d$ → Yokonuma-Hecke algebras

C_d is the Cyclic group of order d.

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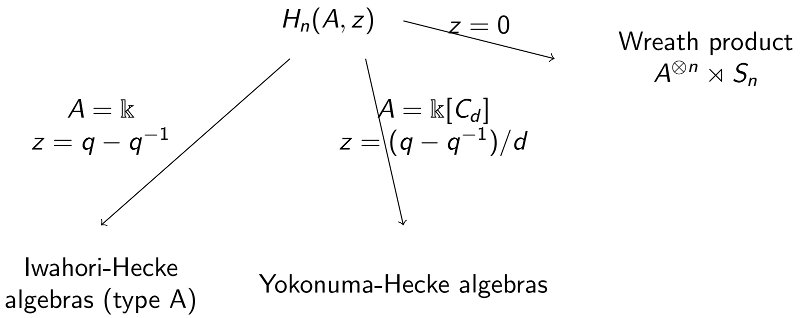

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Affine Frobenius-Hecke algebra $H_n^{\mathrm{aff}}(A, z)$

Fix A, z as before.

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Affine Frobenius-Hecke algebra $H_n^{\text{aff}}(A, z)$

Fix A, z as before.

Affine Frobenius-Hecke algebra

Generated by $H_n(A,z)$, $X_1^{\pm}, \dots, X_n^{\pm}$, with additional relations

$$\begin{aligned}
X_i X_i^{-1} &= X_i^{-1} X_i = 1 \qquad 1 \le i \le n \\
X_i X_j &= X_j X_i \qquad 1 \le i,j \le n \\
T_i X_j &= X_j T_i \qquad 1 \le i \le n-1,\ j \ne i, i+1 \\
T_i X_i T_i &= X_{i+1} \qquad 1 \le i \le n-1 \\
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\end{aligned}$$

Connections with other algebras in the literature

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$H_n^{\mathrm{aff}}(A, z)$

$A = \mathbb{k}$, $z = q - q^{-1}$ → Affine Hecke algebras (type A)

$A = \mathbb{k}[C_d]$, $z = (q - q^{-1})/d$ → Affine Yokonuma Hecke algebras (Chlouveraki-Poulain d'Andecy)

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Connections with other algebras in the literature

$H_n^{\mathrm{aff}}(A, z)$ $\xrightarrow{\text{degen.}}$ Affine wreath product algebras (Savage) $\xrightarrow{\psi = \mathrm{Id},\ A \text{ even}}$ Affinized symmetric algebras (Kleshchev-Muth)

$H_n^{\mathrm{aff}}(A, z)$ $\xrightarrow{A = \Bbbk,\ z = q - q^{-1}}$ Affine Hecke algebras (type A)

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Some Results (R.-Savage)

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Basis Theorem

The map

$$A^{\otimes n} \otimes \Bbbk[X_1^{\pm}, \dots, X_n^{\pm}] \otimes \mathcal{H} \to H_n^{\mathrm{aff}}(A), \qquad \mathbf{a} \otimes p \otimes T_w \mapsto \mathbf{a} p T_w,$$

is an isomorphism of $\Bbbk$-modules.

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Center

If z is not a zero divisor and $K = \ker(t_{1,2}) \subseteq A \otimes A$ is such that $K = K^{S_2}$, then

$$Z(H_n^{\mathrm{aff}}(A, z)) = (Z(A)^{\otimes n})^{S_n} \otimes (\Bbbk[X_1^{\pm}, \dots, X_n^{\pm}])^{S_n}.$$

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Mackey Theorem

There is a version of the Mackey theorem for induction and restriction to parabolic subalgebras.

Cyclotomic Quotients

Let

$$f = X_1^d + \mathbf{a}_{d-1} X_1^{d-1} + \ldots + \mathbf{a}_1 X_1 + \mathbf{a}_0 \in H_n^{\mathrm{aff}}(A, z),$$

be a monic polynomial of degree d in X_1 with coefficients in $Z(A) \otimes 1^{\otimes n-1} \subseteq A^{\otimes n}$, we also assume that $\mathbf{a}_0$ is invertible.

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Cyclotomic Frobenius-Hecke algebra

For a fixed f, we define the corresponding *cyclotomic Frobenius Hecke algebra* to be

$$H_n^f(A, z) := H_n^{\mathrm{aff}}(A, z)/\langle f \rangle,$$

where $\langle f \rangle$ denotes the two-sided ideal generated by f.

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Basis theorem for cyclotomic quotients

The canonical images of the elements

$$\{X_1^{\lambda_1} \cdots X_n^{\lambda_n} \mathbf{a} T_w \mid 0 \leq \lambda_1, \ldots, \lambda_n < d, \ \mathbf{a} \in A^{\otimes n}, w \in S_n\}$$

form a basis of $H_n^f(A, z)$.

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Thank you!