p-DG structures in higher representation theory.

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Outline

- Categorification of cyclotomic ring.
- Quantum $\mathfrak{sl}_2$
- p-DG nilHecke algebra
- Categorification of $V_r \otimes V_s$

Categorification of cyclotomic ring

For a closed 3-manifold M^3 and a prime p, Reshetikhin-Turaev and Witten constructed an invariant $Z(M^3) \in \mathbb{O}_p$ where

$$\mathbb{O}_p = \mathbb{Z}[q]/\Phi_p(q^2)$$
$$\Phi_p(q) = q^{p-1} + \cdots + 1.$$

Categorification of cyclotomic ring

Crane-Frenkel (1994) outlined the program of categorification. The goal is to construct a homology theory of 3-manifolds $\widetilde{Z}(M^3)$ which is functorial under cobordisms.

Thus if W^4 is a cobordism with between M^3 and N^3 then we get a map

$$\widetilde{Z}(W^4)\colon \widetilde{Z}(M^3) \to \widetilde{Z}(N^3)$$

which would hopefully be an invariant of 4-manifolds.

Categorification of cyclotomic ring

The first task is to find a monoidal category whose Grothendieck group is isomorphic to $\mathbb{O}_p$.

Khovanov solved this problem.

- Let $\Bbbk$ be a field of characteristic p.
- Let $H_p = \Bbbk[\partial]/(\partial^p)$ be a graded algebra where the degree of ∂ is 2.
- H_p has a unique simple module (up to isomorphism and grade shift).
- Let L be the 1-dimensional module concentrated in degree 0.

Categorification of cyclotomic ring

This implies that $K_0(H_p\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$ where $[L\langle r\rangle] \mapsto q^r$.

As a module over itself, H_p has a filtration with subquotients $L, L\langle 2\rangle, \ldots, L\langle 2(p-1)\rangle$.

Thus in the Grothendieck group $[H_p] = q^{2(p-1)} + \cdots + 1$.

In order to categorify $\mathbb{O}_p$ we need a category where $H_p \cong 0$.

Categorification of cyclotomic ring

Let $H_p\text{-}\underline{\mathrm{gmod}}$ be the stable category of H_p-modules.

Objects: same as $H_p\text{-}\mathrm{gmod}$.

Morphisms:

$$\mathrm{Hom}_{H_p\text{-}\underline{\mathrm{gmod}}}(M, N) = \mathrm{Hom}_{H_p\text{-}\mathrm{gmod}}(M, N)/I(M, N)$$

where $I(M, N)$ is the subspace of maps which factor through H_p.

Since the identity map of H_p is in the subspace we get the following result due to Khovanov.

Lemma

$K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathbb{O}_p$.

Categorification of cyclotomic ring

Since the representation theory of the small quantum group is defined over $\mathbb{O}_p$, it is important to categorify modules over this ring. Khovanov outlined a procedure to do this.

- Let A be a $\mathbb{Z}$-graded algebra over $\Bbbk$ with a derivation ∂ of degree 2 such that $\partial^p = 0$.
- A is then called a p-DG algebra.

Categorification of cyclotomic ring

- Let N be a p-DG module over A. This means N has a derivation which is compatible with the derivation on A.
- Let $M \in H_p\text{-gmod}$ and $N \in A\text{-pdgmod}$.

Then $M \otimes N \in A\text{-pdgmod}$.

$a \in A$ acts on the second factor and ∂ acts by $\partial \otimes 1 + 1 \otimes \partial$.

This gives a functor

$$H_p\text{-gmod} \times A\text{-pdgmod} \to A\text{-pdgmod}.$$

Categorification of cyclotomic ring

Let $N', N'' \in A\text{-}\mathrm{pdgmod}$.

$$f\colon N' \to N''$$

is said to be nullhomotopic if there exists a map $H\colon N' \to N''$ such that

$$f = \sum_{i=0}^{p-1} \partial^i H \partial^{p-1-i}.$$

Let $A\text{-}\underline{\mathrm{pdgmod}}$ be the homotopy category of $A\text{-}\mathrm{pdgmod}$ where we quotient out by nullhomotopic maps.

Categorification of cyclotomic ring

Khovanov proved that there's a functor

$$H_p\text{-}\underline{\mathrm{gmod}} \times A\text{-}\underline{\mathrm{pdgmod}} \to A\text{-}\underline{\mathrm{pdgmod}}.$$

This endows $K_0(A\text{-}\underline{\mathrm{pdgmod}})$ with the structure of a module over $K_0(H_p\text{-}\underline{\mathrm{gmod}}) \cong \mathbb{O}_p$.

Categorification of cyclotomic ring

Let $f \colon N' \to N''$ be a morphism in $A\text{-}\underline{\mathrm{pdgmod}}$.

f is said to be a quasi-isomorphism if it restricts to an isomorphism in $H_p\text{-}\underline{\mathrm{gmod}}$.

Then we may form the derived category $D(A)$.

Khovanov proved that there's a functor

$$H_p\text{-}\underline{\mathrm{gmod}} \times D(A) \to D(A)$$

which then endows $K_0(D(A))$ with the structure of an $\mathbb{O}_p$-module.

Quantum $\mathfrak{sl}_2$ at root of unity

- Let $\dot{U}$ be $\mathbb{O}_p$-algebra generated by 1_n, $1_{n+2a}E^{(a)}1_n$, $1_{n-2a}F^{(a)}1_n$ for $n \in \mathbb{Z}$, subject to standard relations.
- $u^+ \subset \dot{u} \subset \dot{U}$ where $\dot{u}$ is the small quantum group and $u^+ = \mathbb{O}_p[E]/(E^p)$.
- Let V_l be the Weyl module for $\dot{U}$. It has basis $\{v_0, \dots, v_l\}$.
- $V_r \otimes V_s$ has basis $\{v_b \diamondsuit v_d | 0 \leq b \leq r, 0 \leq d \leq s\}$ given by

$$v_b \diamond v_d = \begin{cases} F^{(d)}E^{(a)}(v_r \otimes v_0) & \text{if } b \leq c \\ E^{(a)}F^{(d)}(v_r \otimes v_0) & \text{if } b \geq c \end{cases}$$

where $a + b = r$ and $c + d = s$.

p-DG nilHecke algebra

Let NH_n be the nilHecke algebra of rank n over $\Bbbk$.

Generators: $y_1, \dots, y_n$ and $\psi_1, \dots, \psi_{n-1}$.

Relations:

- $y_i y_j = y_j y_i$
- $\psi_i^2 = 0$
- $\psi_i \psi_j = \psi_j \psi_i$ for $|i-j| > 1$
- $\psi_i \psi_j \psi_i = \psi_j \psi_i \psi_j$ for $|i-j| = 1$
- $y_i \psi_i - \psi_i y_{i+1} = 1 = \psi_i y_i - y_{i+1} \psi_i$.

NH_n has the structure of a p-DG algebra given by

$$\partial(y_i) = y_i^2 \qquad \partial(\psi_i) = -y_i \psi_i - \psi_i y_{i+1}.$$

p-DG nilHecke algebra

Theorem (Khovanov-Qi)

There is an isomorphism $\bigoplus_{n\geq 0} K_0(\mathcal{D}(NH_n)) \cong u^+$.
Induction and restriction functors descend to multiplication and comultiplication in the Grothendieck group.

Theorem (Elias-Qi)

The Lauda category $\mathcal{U}$ *has a derivation* ∂ *so that* $K_0(\mathcal{D}(\mathcal{U}, \partial)) \cong \dot{u}$.

Theorem (Elias-Qi)

The Khovanov-Lauda-Mackaay-Stosic category $\dot{\mathcal{U}}$ *has a derivation* ∂ *so that* $K_0(\mathcal{D}(\dot{\mathcal{U}}, \partial)) \cong \dot{U}$.

Categorification of $V_r \otimes V_s$

Let $NH_n^l = NH_n/(y_1^l)$ be the cyclotomic nilHecke algebra.

There are induction and restriction functors

$$\mathcal{F}\colon NH_n^l\text{-gmod} \to NH_{n+1}^l\text{-gmod} \qquad \mathcal{E}\colon NH_n^l\text{-gmod} \to NH_{n-1}^l\text{-gmod}$$

giving rise to a categorical $\mathfrak{sl}_2$ (for generic q) action on

$$\bigoplus_{n=0}^{l} NH_n^l\text{-gmod}$$

(Kang-Kashiwara, Chuang-Rouquier, Webster)

Categorification of V_l

NH_n^l is a p-DG algebra and $\mathcal{F}$ and $\mathcal{E}$ are p-DG functors and there is an analogue of the previous result:

Theorem (Elias-Qi, Khovanov-Qi-S)

1. *There's an action of* $(\dot{\mathcal{U}}, \partial)$ *on*

$$\bigoplus_{n=0}^{l} NH_n^l\text{-pdgmod}.$$

2. *When* $l \leq p-1$,

$$\bigoplus_{n=0}^{l} \mathcal{D}(NH_n^l, \partial)$$

categorifies the irreducible Weyl module V_l.

Categorification of $V_r \otimes V_s$

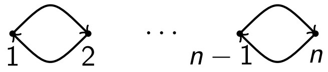
In order to categorify tensor products, we consider certain cyclotomic nilHecke modules introducted by Hu and Mathas.

- Let P^l_n be set of l-tuples whose entries are either 0 or 1 with a total of n ones.
- To $\lambda \in P^l_n$, with ones in positions $a_1, \dots, a_n$, there is monomial $y^\lambda = y_1^{l-a_1} \cdots y_n^{l-a_n}$.
- There's a p-DG NH^l_n-module $G(\lambda) = y^\lambda NH^l_n$.
- The p-DG quiver Schur algebra is

$$S_n(l) = \mathrm{End}_{NH^l_n}\Big(\bigoplus_{\lambda \in P^l_n} G(\lambda)\Big).$$

Categorification of $V_r \otimes V_s$

Example: General l and $n = 1$. In this case $NH_1^l \cong \Bbbk[y]/(y^l)$ and $S_1(l)$ is isomorphic to $A_l^!$ where $A_l^!$ is the quotient of the path algebra of



by the relations

$$(1|2|1) = 0 \qquad (i|i+1|i) = (i|i-1|i)$$

for $i = 2, \ldots, l-1$.

$A_l^!$ has a p-DG structure determined by

$$\partial(i) = 0 \qquad \partial(i+1|i) = 0 \qquad \partial(i|i+1) = (i|i+1|i|i+1)$$

Categorification of $V_r \otimes V_s$

- Fix $r + s = l$, $a + b = r$, $c + d = s$.
- Let $P_n^{r,s} \subset P_n^l$ be tuples of form $\lambda = (0^a 1^b | 0^c 1^d)$.
- To such a λ, there is a certain idempotent $e_\lambda \in NH_n^l$.
- $e_\lambda G(\lambda)$ is a p-DG submodule of $G(\lambda)$.
- Then we have the p-DG algebra

$$S_n(r,s) = \mathrm{End}_{NH_n^l}\Big(\bigoplus_{\lambda \in P_n^{r,s}} e_\lambda G(\lambda) \Big)$$

Categorification of $V_r \otimes V_s$

Theorem (Khovanov-Qi-S)

1. *There's an action of* $(\dot{\mathcal{U}}, \partial)$ *on*

$$S(r,s) := \bigoplus_{n=0}^{r+s} S_n(r,s)\text{-pdgmod}$$

2. *The action of* $\mathcal{D}(\dot{\mathcal{U}}, \partial) \cong \dot{U}$ *on* $\mathcal{D}(S(r,s), \partial)$ *categorifies* $V_r \otimes V_s$*.*