

Traces of tensor product categories

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October 27, 2018

Trace decategorification

The trace (or zeroth Hochschild homology) of a \mathbb{C} -linear additive category \mathcal{C} :

$$\mathrm{Tr}(\mathcal{C}) := \left(\bigoplus_{x \in \mathrm{ob}(\mathcal{C})} \mathrm{End}_{\mathcal{C}}(x) \right) / \mathrm{Span}\{fg - gf\},$$

where f and g run through all pairs of morphisms $f : x \rightarrow y$ and $g : y \rightarrow x$ with $x, y \in \mathrm{Ob}(\mathcal{C})$.

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$\Rightarrow \mathrm{Tr}(\mathcal{C})$ as an algebra.

Relationship between K_0 and Tr

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Have a *Chern character map*

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Additional advantage: trace is invariant under taking Karoubi envelope.

Categorified quantum groups

[Khovanov-Lauda] and [Rouquier] independently constructed categories $\mathbf{U}(\mathfrak{g})$ such that

$$K_0(\mathbf{U}(\mathfrak{g})) \cong \dot{\mathcal{U}}_q(\mathfrak{g})$$

where $\dot{\mathcal{U}}_q(\mathfrak{g})$ - idempotent form of quantum group associated to \mathfrak{g} .

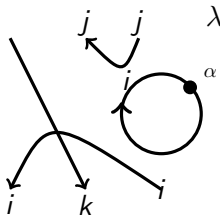
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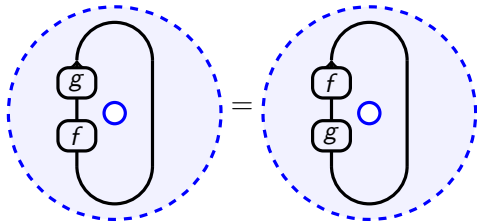
Morphisms given by *KL diagrams*:



modulo relations of the *quiver Hecke algebra*.

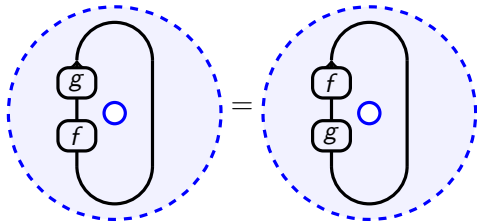
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Denote by brackets an element's image in trace, e.g.

$$\left[\begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \right]$$

Trace of categorified quantum groups

[Beliakov-Habiro-Lauda-Webster]: for \mathfrak{g} simply laced,

$$\mathrm{Tr}(\mathbf{U}^*(\mathfrak{g})) \cong \dot{\mathcal{U}}(\mathfrak{g}[t]).$$

$\dot{\mathcal{U}}(\mathfrak{g}[t])$ - idempotent form of current algebra.

$$(E_i \otimes t^r)1_\lambda \longmapsto \left[\begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} \begin{array}{c} \lambda \\ r \\ i \end{array} \right], \quad (F_j \otimes t^s)1_\lambda \longmapsto \left[\begin{array}{c} \downarrow \\ \bullet \\ \uparrow \end{array} \begin{array}{c} \lambda \\ s \\ j \end{array} \right].$$

Categorifying modules

Irreducible $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules \leftrightarrow Cyclotomic quotient
 $V(\lambda)$ K_0 \mathbf{U}^λ

$$\langle i, \lambda \rangle \bullet \cdots = 0$$

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[BHLW] \mathfrak{g} simply laced:

$$\mathrm{Tr}(\mathbf{U}^{\lambda,*}) = W(\lambda) \text{ (local Weyl module for } \dot{\mathcal{U}}(\mathfrak{g}[t]).$$

Deformed cyclotomic quotient $\mapsto \mathbb{W}(\lambda)$ (global Weyl module)

Categorifying tensor products

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a sequence of dominant weights.

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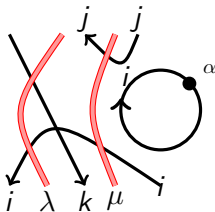
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Can be used to prove nondegeneracy of categorified quantum groups for symmetrizable root data.

Stendhal diagrams

Morphisms in \mathcal{T} are given by *Stendhal diagrams*.



Red strands labeled by dominant weights.

We prove:

Theorem

For \mathfrak{g} simply laced, there is an algebra isomorphism

$$\mathrm{Tr}(\mathcal{T}^*(\underline{\lambda})) \longrightarrow W(\underline{\lambda}) = W(\lambda_1) \otimes \dots \otimes W(\lambda_n)$$

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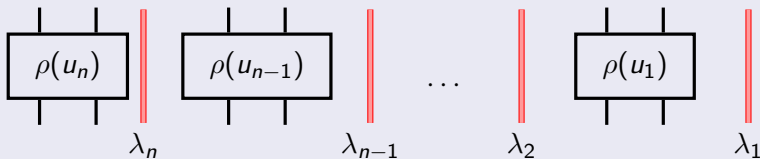
The trace of a deformed version is isomorphic to $\mathbb{W}(\underline{\lambda})$.

Constructing the map

Lemma

The map $W(\underline{\lambda}) \rightarrow \text{Tr}(\mathcal{T}^*(\underline{\lambda}))$

$$u_n(\cdots u_2((u_1 w_{\lambda_1}) \otimes w_{\lambda_2}) \otimes \cdots \otimes w_{\lambda_n}) \longmapsto$$



is an algebra homomorphism (ρ is the isomorphism from BHLW).

Surjectivity

We show that $\text{Tr}(\mathcal{T}^*(\underline{\lambda}))$ is spanned by Stendhal diagrams with no red-black crossings:

The diagram shows an equality between two Stendhal diagrams. On the left, a diagram is enclosed in large square brackets. It features two black lines crossing: one from the bottom-left to the top-right, and another from the top-left to the bottom-right. The bottom-left input is labeled i and the bottom-right input is labeled j . A red curved line, representing a crossing, is drawn over the black lines. A horizontal dotted line is drawn across the middle of the diagram. On the right, the same diagram is shown with an equals sign. It consists of two parts enclosed in large square brackets. The first part is a crossing of two black lines, with the bottom-left input labeled i and the bottom-right input labeled j . A black dot is placed on the top-right line of the crossing. The second part is a vertical red line, with the label λ positioned below it.

These are clearly in the image of the map.

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Upper semicontinuity under deformation:

$$\dim \text{ at "special point" } \geq \dim \text{ at generic point}$$

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Deform category so that special point is $\text{Tr}(\mathcal{T}^*(\underline{\lambda}))$, and generic point has a known dimension.

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- [Webster] *Unfurling Khovanov-Lauda-Rouquier algebras*. 2016