An elliptic Schur-Weyl construction of the rectangular representation of the DAHA

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Special Session on Combinatorial and Categorical Aspects of Representation Theory
AMS Fall Western Sectional Meeting

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mathematical goals

LHS: $\mathcal{D}_q(\frac{G}{G})$-mod

Understand the category of strongly equivariant $\mathcal{D}_q(G)$-modules, where $\mathcal{D}_q(G)$ is the algebra of quantum differential operators on G, $G = \mathrm{GL}(N), \mathrm{SL}(N)$

How? By using Jordan's functor (Schur-Weyl)

$$\mathcal{F}^d : \mathcal{D}_q(\frac{G}{G})\text{-mod} \to \text{DAHA-mod}$$
$$\mathcal{M} \mapsto (V^{\otimes d} \bigotimes \mathcal{M})^{\mathrm{inv}}$$

where $V =$ the N-dimensional defining representation of G

RHS: DAHA-mod

DAHA $=$ double affine Hecke algebra

goal of talk

Define family of functors (Jordan 0805.2766)

$$\mathcal{F}^d : \mathcal{D}_q(\frac{G}{G})\text{-mod} \to \text{DAHA -mod}$$

and describe $\mathcal{F}^d$(basic) for $G = \mathrm{GL}(N)$ or $\mathrm{SL}(N)$

$d = kN$

basic $\mapsto$ rectangular

$$\mathcal{O}_q(G) \mapsto L(k^N)$$

Combinatorics

Combinatorial analysis of $\mathcal{F}^{kN}(\mathcal{O}_q(G))$ and $L(k^N)$

Combinatorics $\leftsquigarrow$ Multiplicity-free phenomena

Pieri Rule (special case)

$$s_{\square} s_\lambda = h_1 s_\lambda = e_1 s_\lambda = \sum_{\mu=\lambda+\square} s_\mu$$

$G = \mathrm{GL}(N)$ or $\mathrm{SL}(N)$ $\quad \mathfrak{g} = \mathfrak{gl}(N)$ or $\mathfrak{sl}(N)$

Defining representation: $V = V(\varepsilon_1) = \mathbb{C}^N \circlearrowleft G \quad$ (or $\mathfrak{g}$ or $\mathcal{U}_q(\mathfrak{g})$)

Weights of V are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$

$$V \bigotimes V(\lambda) \simeq \bigoplus_{\substack{1 \leqslant i \leqslant N \\ \lambda+\varepsilon_i \in P^+}} V(\lambda + \varepsilon_i)$$

$\dim \mathrm{Hom}_{\mathfrak{g}}(V(\mu), V \otimes V(\lambda)) = 1$ or 0

Combinatorics $\leftsquigarrow$ Multiplicity-free phenomena

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), V \bigotimes V(\lambda)) = 1 \text{ or } 0$$

Iterate to get basis of $\operatorname{Hom}_{\mathfrak{g}}(V(\mu), V^{\otimes d} \bigotimes V(\lambda))$

Basis arises as eigenbasis of commuting operators built out of Casimir (ribbon element, double braiding, R-matrix).

$V^{\otimes d-1} \otimes V \otimes V(\lambda)$ $\qquad$ $V^{\otimes d-2} \otimes V \otimes V \otimes V(\lambda)$ $\qquad$ $V^{\otimes d} \otimes V(\lambda)$

Case $\mu = \lambda$

$\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{g})}(V(\lambda) \otimes \det^{d/N}, V^{\otimes d} \bigotimes V(\lambda)) \simeq (V^{\otimes d} \bigotimes V(\lambda) \otimes V(\lambda)^*)^{\mathrm{inv}}$

AHA = affine Hecke algebra (of type *A*)

$$\left(V^{\otimes d} \bigotimes V(\lambda) \otimes V(\lambda)^*\right)^{\mathrm{inv}}$$

is an AHA_d representation
Arakawa–Suzuki 9710037, Arakawa–Suzuki–Tsuchiya 9710031

DAHA = double affine Hecke algebra (of type *SL* or *GL*)

$$\bigoplus_{\lambda \in P^+} \left(V^{\otimes d} \bigotimes V(\lambda) \otimes V(\lambda)^*\right)^{\mathrm{inv}}$$

is a DAHA_d representation.

$d = kN$

DAHA, $d = kN$

$$\left(V^{\otimes d} \bigotimes \mathcal{O}_q(G)\right)^{\text{inv}} = \left(V^{\otimes d} \bigotimes \left(\bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^* \right)\right)^{\text{inv}}$$

$$\left(V^{\otimes d} \bigotimes \mathcal{O}_q(G)\right)^{\text{inv}} = \mathcal{F}^d(\mathcal{O}_q(G))$$

$$\left(V^{\otimes d} \bigotimes \mathcal{M}\right)^{\text{inv}} = \mathcal{F}^d(\mathcal{M}) \text{ Jordan's functor}$$

Theorem (Jordan-V 1708.06024)

The DAHA_d *representation*

$$\mathcal{F}^d(\mathcal{O}_q(G)) \simeq L(k^N)$$

$d = kN$

DAHA

The DAHA_d representation

$$\mathcal{F}^d(\mathcal{O}_q(G)) = \left(V^{\otimes d} \bigotimes \Big(\bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^* \Big) \right)^{\mathrm{inv}} \simeq L(k^N)$$

$\mathcal{U}_q(\mathfrak{g})$ structure on $\mathcal{O}_q(G)$ (Peter-Weyl type theorem)

$$\mathcal{O}_q(G) \simeq \bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^*$$

How do we recognize which DAHA representation?
DAHA $\supseteq \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_d^{\pm 1}] =: \mathcal{Y}$.

RHS and LHS each have Y-weight-basis, i.e. $\mathrm{Res}_{\mathcal{Y}}$ is multiplicity-free.
When action of Y_i on basis is "nice," can recover module structure.

DAHA action on $\mathcal{F}(\mathcal{M})$, $\mathcal{M}$ is a $\mathcal{D}_q(G)$-module

$$Y_1 = (R_{12}R_{21})_1 = (\rho_V \otimes \mathrm{id})(R_{12}R_{21}) \in \mathrm{End}(V) \bigotimes \mathcal{U}_q(\mathfrak{g})$$

$$X_1 = (R_{12}R_{21})_2$$

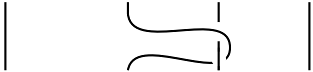
Build on	Calaque-Enriquez-Etingof	0702670
	Arakawa-Suzuki	9710037
	Arakawa-Suzuki-Tsuchiya	9710031
	Lyubashenko-Majid	

DAHA action (Jordan 0805.2766)

$T_i \sim R$ matrix

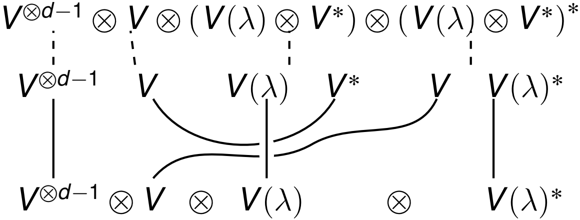
$$V \otimes \cdots \otimes V \otimes V \otimes \cdots \otimes \mathcal{M}$$

Y_1 double braiding



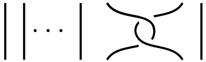
$$V^{\otimes d-1} \otimes V \otimes V(\lambda) \otimes V(\lambda)^*$$

X_1



DAHA action (Jordan 0805.2766)

Since we are lucky, it suffices to calculate how the Y_i act.
$Y_i = T_{i-1} Y_{i-1} T_{i-1}$

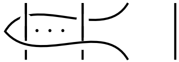


$V^{\otimes d-1} \otimes V \otimes V(\lambda) \otimes V(\lambda)^*$

Y_1

$V^{\otimes d-2} \otimes V \otimes V \otimes V(\lambda) \otimes V(\lambda)^*$

Y_2



$V^{\otimes d} \otimes V(\lambda) \otimes V(\lambda)^*$

Y_d

Luck here means multiplicity-free phenomenon AND we can “ignore” the $V(\lambda)^*$

$\mathfrak{sl}(2)$ $N = 2$ combinatorics

Pieri rule

$m = \lambda \in P^+ = \mathbb{Z}_{\geqslant 0}$

$$V \bigotimes V(m) \simeq V(m+1) \bigoplus V(m-1)$$

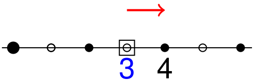
if $m \geqslant 1$

$\mathfrak{sl}(2) \quad N = 2 \qquad d = 4 = 2 \cdot 2 \quad m = 3$

Iterated

$$V \otimes V \otimes V \otimes V \bigotimes V(3) \qquad \otimes V(3)^*$$

$$V \otimes V \otimes V \otimes V(4) \qquad \otimes V(3)^*$$



$$V \otimes V \otimes V(5) \qquad \otimes V(3)^*$$

$$V \otimes V(4) \qquad \otimes V(3)^*$$

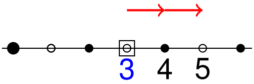
$$V(3) \qquad \otimes V(3)^*$$

$\mathfrak{sl}(2) \quad N = 2 \qquad d = 4 = 2 \cdot 2 \quad m = 3$

$V \otimes V \otimes V \otimes V \bigotimes V(3) \qquad \otimes V(3)^*$

$V \otimes V \otimes V \otimes V(4) \qquad \otimes V(3)^*$

$V \otimes V \otimes V(5) \qquad \otimes V(3)^*$



$V \otimes V(4) \qquad \otimes V(3)^*$

$V(3) \qquad \otimes V(3)^*$

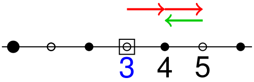
$\mathfrak{sl}(2)$ $\quad N = 2 \qquad d = 4 = 2 \cdot 2 \quad m = 3$

$V \otimes V \otimes V \otimes V \bigotimes V(3) \qquad \otimes V(3)^*$

$V \otimes V \otimes V \otimes V(4) \qquad \otimes V(3)^*$

$V \otimes V \otimes V(5) \qquad \otimes V(3)^*$

$V \otimes V(4) \qquad \otimes V(3)^*$



$V(3) \qquad \otimes V(3)^*$

$\mathfrak{sl}(2) \quad N = 2 \qquad d = 4 = 2 \cdot 2 \quad m = 3$

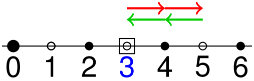
$$V \otimes V \otimes V \otimes V \bigotimes V(3) \qquad \otimes V(3)^*$$

$$V \otimes V \otimes V \otimes V(4) \qquad \otimes V(3)^*$$

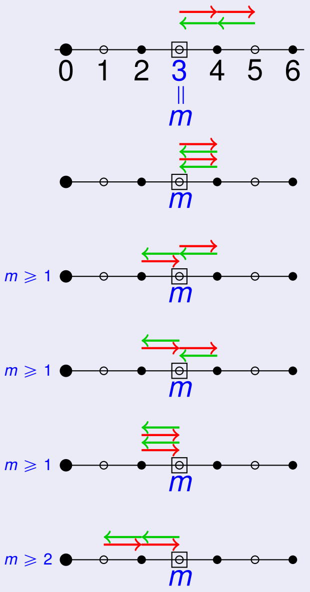
$$V \otimes V \otimes V(5) \qquad \otimes V(3)^*$$

$$V \otimes V(4) \qquad \otimes V(3)^*$$

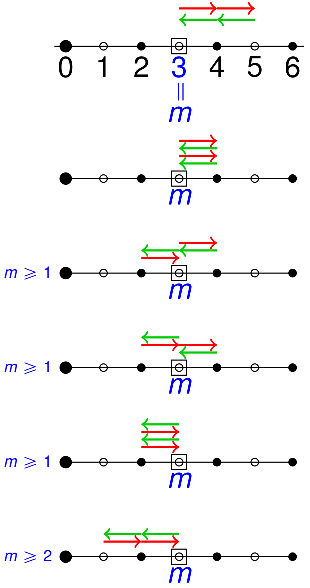
$$V(3) \qquad \otimes V(3)^*$$



$\mathfrak{sl}(2)$, $N = 2$, $d = 4 = 2 \cdot 2 = kN$ basis



$\mathfrak{sl}(2)$, $N = 2$, $d = 4 = 2 \cdot 2 = kN$ skew tableaux



			1	2
3	4			

m			1	3
2	4	$_*\omega$		

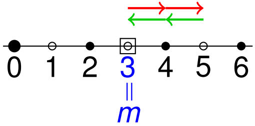
m			1	4
2	3	$_*\omega$		

m			2	3
1	4	$_*\omega$		

m			2	4
1	3	$_*\omega$		

m			3	4
1	2	$_*\omega$		

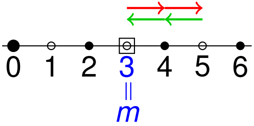
$\mathfrak{sl}(2)$, $N = 2$, $d = 4 = 2 \cdot 2 = kN$, d-periodic



−7	−6	−3	−2	1	2	5	6
0	3	4	7	8	11	12	15

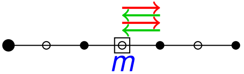
−2	1
7	8

$\mathfrak{sl}(2)$, $N = 2$, $d = 4 = 2 \cdot 2 = kN$ d-periodic



			1	2
3	4			

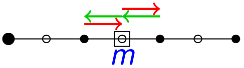
-6	-3	-2	1	2
3	4	7	8	11



			1	3
2	4			

-5	-3	-1	1	3
2	4	6	8	10

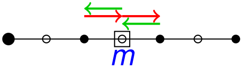
$m \geqslant 1$



			1	4
2	3			

-4	-3	0	1	4
2	3	6	7	10

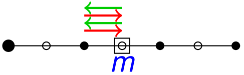
$m \geqslant 1$



			2	3
1	4			

-5	-2	-1	2	3
1	4	5	8	9

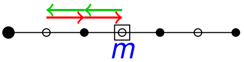
$m \geqslant 1$



			2	4
1	3			

-4	-2	0	2	4
1	3	5	7	9

$m \geqslant 2$

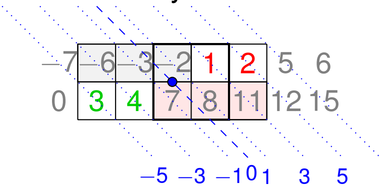


			3	4
1	2			

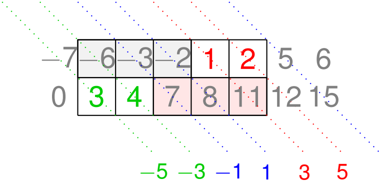
-4	-1	0	3	4
1	2	5	6	9

DAHA action on $L(k^N)$ Y_i on weight basis

DAHA already knows how to act on periodic tableaux



−2	1
7	8



diagonals

spectrum

Y_1	Y_2	Y_3	Y_4
q^3	q^5	q^{-5}	q^{-3}

Casimir ν

Y_1

$V^{\otimes d-1} \otimes V \otimes V(\lambda) \otimes V(\lambda)^*$

on inv $\quad \Delta(\nu) = R_{21} R_{12} (\nu^{-1} \otimes \nu^{-1})$

$$\frac{\mathrm{Cas}(m+1)}{\mathrm{Cas}(m)\,\mathrm{Cas}(1)} = q^{\mathrm{diag}\,\boxed{1}}$$

$\mathfrak{sl}(N)$

$$q^{i-1} \cdot t^{\langle \gamma+2\rho,\gamma\rangle - \langle \beta+2\rho,\beta\rangle - \langle \omega_1+2\rho,\omega_1\rangle} = q^{\mathrm{diag}\,\boxed{i}}$$

$$\lambda \to \cdots \beta \xrightarrow{i^{th}\text{step}} \gamma \cdots \to \lambda$$

Y_i spectrum

DAHA action, $G = \mathrm{GL}(N)$

$L :=$ span$\{d$-periodic skew tableaux $\}$
Basis = Y-weight basis of L

[Ram, Suzuki-V] Theory of Y-semisimple/ *calibrated* DAHA$_d$-modules
$\Longrightarrow$

$$L \simeq L(k^N)$$

$$\|$$

$$L\left(\underbrace{\boxed{\quad\quad N\left\{\vphantom{\rule{0pt}{3em}}\right.}}_{k}\right)$$

$d = kN$, $G = \mathrm{GL}(N)$

$H_d^{\mathrm{aff}} \twoheadrightarrow H_d \circlearrowright$ N  k

$\mathrm{Ind}_{\mathrm{AHA}}^{\mathrm{DAHA}}$ N  k $=: \; M(k^N) \twoheadrightarrow L(k^N)$

unique
simple
quotient

Why is $\left(V^{\otimes d} \bigotimes \mathcal{O}_q(G)\right)^{\mathrm{inv}} \simeq L(k^N)$?
Same $\mathcal{Y}$ spectrum, $\exists\, \mathcal{Y}$-weight basis

Future: other $\mathcal{M}$

Thank you

$\mathcal{D}_q(G)$

	$\mathcal{D}_q(G)$	$\mathcal{O}_q(G)$	DAHA	
			$X^{\pm}$	$Y^{\pm}$
	$\mathcal{D}(G)$	$\mathcal{O}(G)$		
	$\mathcal{D}(\mathfrak{g})$	$\mathbb{C}[\mathfrak{g}]$	RCA	
			x	y
			creation	Dunkl
			acts freely	
Toy	$\mathbb{C}[x, \partial_x]$	$\mathbb{C}[x]$	x	∂_x

$\mathcal{O}_q(G)$

Today's $\mathcal{D}_q(G)$-module is $\mathcal{M} = \mathcal{O}_q(G)$
$=$ quantum deformation of polynomial functions on G
$\simeq$ "locally finite subalgebra" $\mathcal{U}_q(\mathfrak{g})^{\mathrm{locfin}}$
$\simeq$ algebra of matrix coefficients

$\mathcal{O}_q(G)$ quantizes the Semenov-Tian-Shansky bracket, so is conjugation equivariant (vs Sklyanin bracket L/R)

$$\mathcal{D}_q(G) = \mathcal{O}_q(G) \widetilde{\otimes} \mathcal{O}_q(G)$$

carries 3 actions of $\mathcal{U}_q(\mathfrak{g})^{\mathrm{locfin}}$ — ad, R, L

$\mathcal{U}_q(\mathfrak{g})^{\mathrm{locfin}}$ structure

$$\mathcal{M} = \mathcal{O}_q(G) \simeq \bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^*$$

(Peter-Weyl type theorem)