Categorical Bernstein Operators

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Theorem (Zelevinsky)

$$B_{\lambda_1} \ldots B_{\lambda_n}(1) = s_\lambda$$

$$B^*_{\lambda_n} \ldots B^*_{\lambda_1}(s_\lambda) = 1$$

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$$\mathrm{P} \rightsquigarrow \ \mathrm{n{+}1} \ \Big\uparrow \ \mathrm{n} \ \rightsquigarrow Ind_n^{n+1} : \mathbb{C}[S_n]\text{-mod} \to \mathbb{C}[S_{n+1}]\text{-mod}$$

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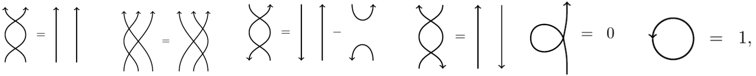
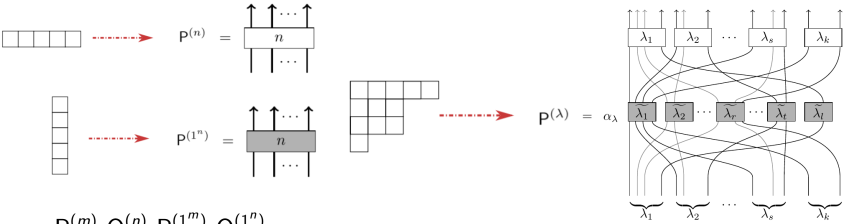
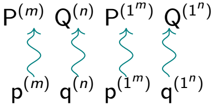
Khovanov's Heisenberg Category $\mathcal{H}$: a monoidal, idempotent complete category whose object are generated by P, Q, and morphisms by:

subject to the relations:







Categorical Fock Space

$$V_{Fock} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}[S_n] - mod$$

Induction and Restriction

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$$B_1 = \cdots - p^{(4)}q^{(1^3)} + p^{(3)}q^{(1^2)} - p^{(2)}q + p$$

$$\rightsquigarrow \cdots \to \mathsf{P}^{(4)}\mathsf{Q}^{(1^3)} \to \mathsf{P}^{(3)}\mathsf{Q}^{(1^2)} \to \mathsf{P}^{(2)}\mathsf{Q} \to \underline{\mathsf{P}} := \mathsf{B}_1$$

The **categorical Bernstein operators** are the chain complexes

$$\mathsf{B}_a := \ldots \mathsf{P}^{(a+n)}\mathsf{Q}^{(1^n)} \to \mathsf{P}^{(a+n-1)}\mathsf{Q}^{(1^{n-1})} \to \ldots \qquad \in \mathcal{K}^-(\mathcal{H}) \qquad a \in \mathbb{Z}$$

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Theorem (G)

Given $\lambda_1 \geq \dots \geq \lambda_n$ then

$$B_{\lambda_1} \otimes \dots \otimes B_{\lambda_n}(\mathbb{1}) = \mathsf{P}^{\lambda_1,\dots,\lambda_n}(\mathbb{1})$$

$$B^*_{\lambda_n} \otimes \dots \otimes B^*_{\lambda_1}\mathsf{P}^{\lambda_1,\dots,\lambda_n}(\mathbb{1}) = \mathbb{1}$$

Moreover the Bernstein operators satisfy the commutation relations:

$$B_{a-1}B_b + B_{b-1}B_a = 0 \qquad B^*_{a+1}B^*_b + B^*_{b+1}B^*_a = 0 \qquad B_{a+1}B^*_{b+1} + B^*_b B^*_a = \delta_{a,b}$$

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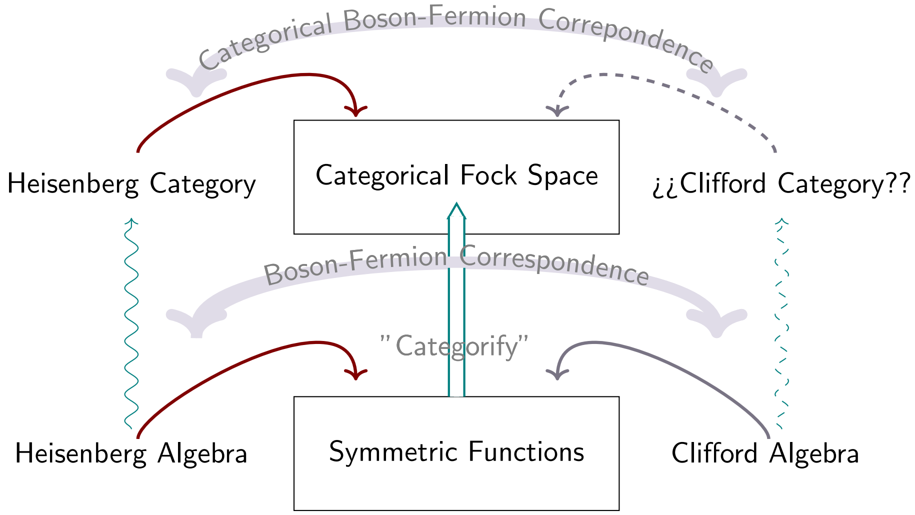

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Theorem (Kac–Boson-Fermion correspondence)

The Clifford algebra acts on $\bigoplus_{c\in\mathbb{Z}} t^c Sym$ *and*

$$\Psi_i(t^c f) = t^{c+1} B_{i-c-1}(f)$$

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Theorem (G)

The functors $\Psi_i = \mathcal{Q}B_{i-a-1}$ and $\Psi_i^* = \mathcal{Q}B_{i-a}^*$ satisfy categorical analogues of the Clifford algebra relations and categorify the Boson-Fermion correspondence.

The Clifford algebra has generators Ψ_i, Ψ_i^* for $i \in \mathbb{Z}$ and commutation relations $\{\Psi_i, \Psi_j\} = 0$ and $\{\Psi_i^*, \Psi_j^*\} = 0$ and $\{\Psi_i, \Psi_j^*\} = \delta_{i,j}$.

Theorem (Kac–Boson-Fermion correspondence)

The Clifford algebra acts on $\bigoplus_{c \in \mathbb{Z}} t^c Sym$ and

$$\Psi_i(t^c f) = t^{c+1} B_{i-c-1}(f)$$

$$\Psi_i^*(t^c f) = t^{c-1} B_{i-c}^*(f)$$

Cautis and Sussan conjectured a categorical version of Kac's theorem, i.e. a categorical Boson-Fermion correspondence.

Theorem (G)

The functors $\Psi_i = \mathcal{Q}B_{i-a-1}$ and $\Psi_i^* = \mathcal{Q}B_{i-a}^*$ satisfy categorical analogues of the Clifford algebra relations and categorify the Boson-Fermion correspondence.

Thank you for listening!

References:

- Cautis, S., and Sussan, J. On a categorical Boson-Fermion correspondance, Communications in Mathematical Physics,(2015) 336, 649
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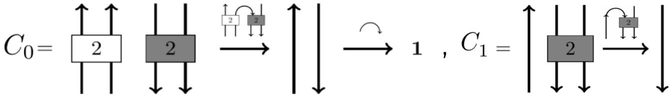
- Frenkel I., Penkov I., Serganova V., A categorification of the boson-fermion correspondence via representation theory of $sl(\infty)$, Comm.Math.Phys 341:911-931.
- Khovanov, M. Heisenberg algebra and a graphical calculus, Fundamenta Mathematicae 225, (2014), 169-210
- Tian, Yin, Towards a categorical Boson-Fermion correspondence, arXiv:1710.11579

....all the Heisenbergs......

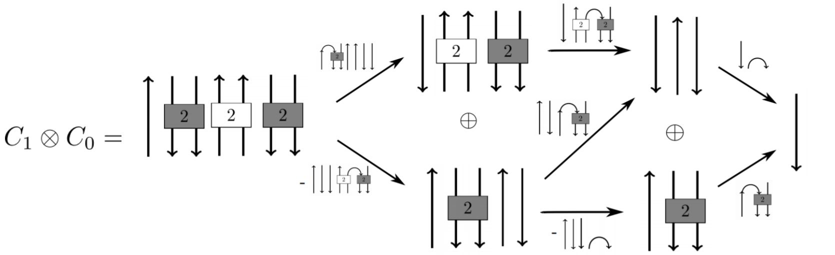
Example

Suppose $i = 0$ and $Q^m = 0$ for $m \geq 3$. Then $\Psi_0 \otimes \Psi_0 \cong 0$ $\Leftrightarrow C_{n+1} \otimes C_n \cong 0$ for all n.

Let $n = 0$ then



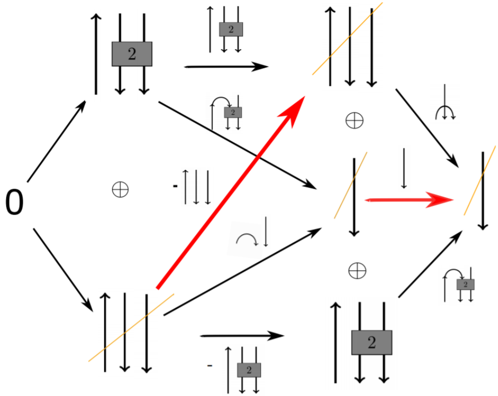
so:



Example

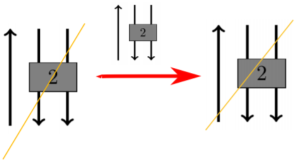
Apply isomorphism QPQ ≅ PQQ ⊕ Q then:

$C_1 \otimes C_0 \cong$



Example

$$C_1 \otimes C_0 \cong \quad 0 \longrightarrow$$



$$\longrightarrow 0$$

$$\cong \quad 0$$