The classification of a(2)-finite Coxeter groups

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Coxeter groups

Definition

A Coxeter system is a pair (W, S), in which W is a group given by the presentation

$$\langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

and the $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfy $m_{ii} = 1$ and $m_{ij} = m_{ji}$.
(If $m_{ij} = \infty$, we omit the corresponding relation.)

The group W above is known as a Coxeter group. Well known examples of Coxeter groups include the finite symmetric and dihedral groups. It turns out that the generators s_i have order 2 and that m_{ij} is the order of the product $s_i s_j$.

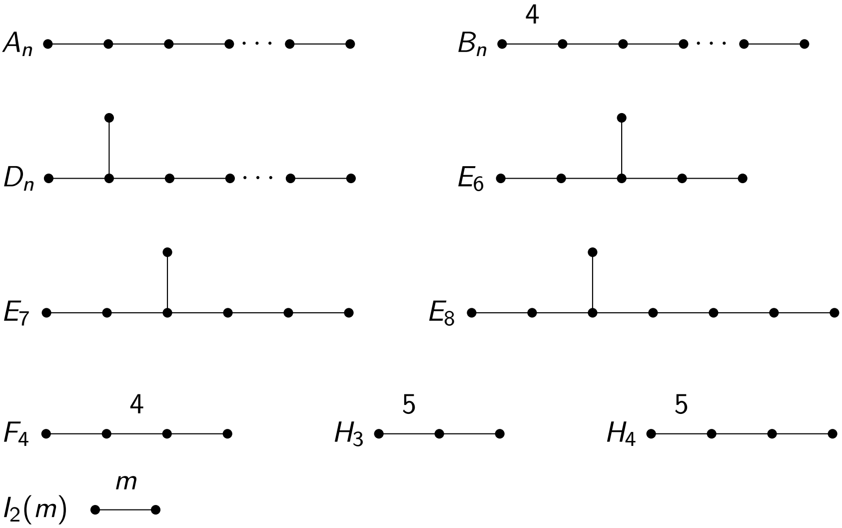
A good way to encode the information given in the presentation of a Coxeter group is by means of a graph.

Definition

Let (W, S) be a Coxeter system. The Coxeter graph, Γ, of (W, S) is a graph whose vertices are indexed by S. Two vertices s_i and s_j are connected by an edge if $m_{ij} > 2$. If we have $m_{ij} > 3$, then we label the edge by the integer $m_{ij} = m_{ji}$.

Theorem (Classification of finite Coxeter groups)

A Coxeter group is finite if and only if the connected components of its Coxeter graph are finite in number and appear in the list in the next figure.


A_n
4
B_n
D_n
E_6
E_7
E_8
4
F_4
5
H_3
5
H_4
m
$I_2(m)$

Reduced expressions

Every element of a Coxeter group W can be expressed as a word in the generators S.

For a given element $w \in W$, the length of w, $\ell(w)$, is defined to be the minimal integer k for which w has an expression as a word of length k in the alphabet S.

Minimal words of this type are called reduced expressions for w. We call an element $w \in W$ rigid if it has a unique reduced expression.

Provided that $i \neq j$, the defining relation $(s_i s_j)^{m_{ij}} = 1$ of a Coxeter group may be written in the homogeneous form

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}}.$$

A relation of this form is called a braid relation. We call the braid relation short if $m_{ij} = 2$; in other words, if it is of the form $s_i s_j = s_j s_i$ where i and j correspond to nonadjacent nodes in the Coxeter graph.

Theorem (Matsumoto's Theorem)

Let W be a Coxeter group and let $w \in W$. Any reduced expression for w may be transformed into any other via a finite sequence of braid relations.

Fully commutative elements

Two words in the Coxeter generators S are said to be commutation equivalent if one can pass from one to the other using a (finite) sequence of short braid relations.

Definition

Let W be a Coxeter group and let $w \in W$. We call w fully commutative if all reduced expressions for w are commutation equivalent to each other.
The set of fully commutative elements of W is denoted by W_c.

Rigid elements, such as $x = s_1 s_2 s_3$, are vacuously fully commutative.
It turns out that the number of fully commutative elements in the symmetric group S_n is counted by the Catalan numbers.

Heaps

A heap over a graph can be thought of as a certain partially ordered set whose elements are labelled by the vertices of the graph.

Definition

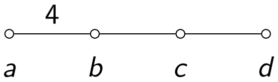
A heap is a function $\varepsilon : E \to \Gamma$, where E is a poset and Γ is a graph, satisfying the following two conditions.

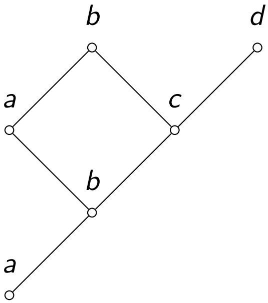
1. The inverse images of each vertex $\varepsilon^{-1}(a)$ and each edge $\varepsilon^{-1}(\{a, b\})$ are chains in E (called vertex chains and edge chains, respectively).
2. The partial order $\leq$ on E is the smallest partial order in which these are chains.

The heaps over a fixed Γ can be made into a category. Here, the isomorphisms are what you would expect them to be: isomorphisms of partially ordered sets that respect the labellings.

For our purposes today, the sets E and Γ will always be finite. In this case, it is convenient to depict a heap (up to isomorphism) by labelling the vertices of the Hasse diagram of E with the elements of Γ.

The next diagram shows a heap E with six elements, over the Coxeter graph of type B_4.


4
a
b
c
d


b
d
a
c
b
a

Words from heaps

Let (W, S) be a Coxeter system. Any finite heap E over the Coxeter graph of (W, S) gives rise to an element of S^* (i.e., a word in S).

To produce a word from E, first choose a refinement of the partial order $\leq$ on E to a total order, and write

$$E = \{e_1, e_2, \ldots, e_k\}$$

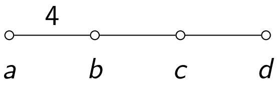
so that $e_1 < e_2 < \cdots < e_k$ according to this total order. The word of E is then the corresponding sequence of labels,

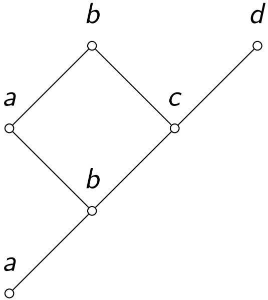
$$\varepsilon(e_1)\varepsilon(e_2)\cdots\varepsilon(e_k).$$

It can be shown that choosing a different refinement of $\leq$ will result in a word that is commutation equivalent to the original word. This means that the two words will be expressions for the same element of W.

More generally, this map from heaps to words can be shown to induce a bijection between isomorphism classes of heaps and commutation classes of words.

Using this, we can talk about the heap of a reduced expression, and if w is fully commutative, we can talk about the heap of w, meaning the heap of a(ny) reduced expression of w.





The heap of the fully commutative element
$abacbd = abcabd = abcadb = abacdb$
in the Coxeter group $W(B_4)$.

Lusztig's a-function

For any Coxeter system (W, S), there is a corresponding Hecke algebra, $\mathcal{H}(W)$. This is an associative algebra over $\mathbb{Z}[q, q^{-1}]$ with basis $\{T_w : w \in W\}$.

Here, q is an indeterminate, and the ring structure of $\mathcal{H}(W)$ is determined by the conditions

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w); \\ qT_{sw} + (q-1)T_w & \text{otherwise.} \end{cases}$$

Using this one weird trick [1], it is possible to replace the basis $\{T_w : w \in W\}$ by another basis with more interesting properties: the Kazhdan–Lusztig basis $\{C'_w : w \in W\}$. This involves enlarging the ground ring to contain a square root, v, of q.

The structure constants $g_{x,y,z}$ of the Kazhdan–Lusztig basis, which are defined by

$$C'_x C'_y = \sum_{z \in W} g_{x,y,z} C'_z,$$

are Laurent polynomials in $\mathbb{Z}[v, v^{-1}]$ (proved to lie in $\mathbb{N}[v, v^{-1}]$ by Elias and Williamson).

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[1] The "one weird trick" corresponds to Stanley's notion of P-kernels, or to Du's notion of IC bases.

For any Coxeter group W, Lusztig defined a function $a : W \to \mathbb{Z}$, where $a(w)$ is defined to be the maximum possible degree of the Laurent polynomial $g_{x,y,w}$ as x and y range over W.

The a-function is useful in representation theory because it is constant on the two-sided Kazhdan–Lusztig cells, which form a partition of W. This makes the a-function relevant to the study of the asymptotic Hecke algebra J.

In general, it is difficult to compute the values of $a(w)$ in a non-recursive way.

Fully commutative elements are relevant to the study of the a-function because of the following result.

Theorem

Let W be a Coxeter group and let $w \in W$.

1. *The element w is the identity if and only if $a(w) = 0$.*
2. *The element w is rigid if and only if $a(w) \leq 1$.*
3. *If $a(w) \leq 2$, then w is fully commutative.*

Definition

Let k be a natural number. We say that a Coxeter group W is $a(k)$-finite if W has only finitely many elements with a-value equal to k.

It follows from the previous theorem that every Coxeter group is $a(0)$-finite, and that W is $a(1)$-finite if and only if W has finitely many rigid elements. It is not too difficult to determine when this latter situation occurs.

Theorem

Let (W, S) be a Coxeter system with Coxeter graph Γ, and assume that Γ is connected. Then W is $a(1)$-finite if and only if both

1. *Γ contains no circuits, and*
2. *Γ has at most one edge with a label strictly bigger than 3.*

(We will often assume for simplicity that the Coxeter graph of W is finite and connected. General properties can typically be easily understood in terms of the answer in the connected case.)

In this context, a natural question is to try to classify the $a(2)$-finite Coxeter groups. This is much harder, but still combinatorially tractable because elements of a-value 2 are fully commutative, and fully commutative elements can be understood in terms of their heaps.

More specifically, the a-value of a fully commutative element may sometimes be computed in terms of the antichains of the corresponding heap. Recall that an antichain in a partially ordered set E is a subset F of E such that no two distinct elements of F are comparable.

Conjecture

Let W be a Coxeter group, let $w \in W_c$, and let $n(w)$ be the size of the longest antichain in the heap of w. Then $a(w)$ is equal to $n(w)$.

It is not hard to prove that $n(w) \leq a(w)$.

Equality is known to hold in certain classes of groups: J.-Y. Shi proved that equality holds if W is a finite or affine Weyl group, and I proved that equality holds if W is a "star reducible" Coxeter group.

Main result

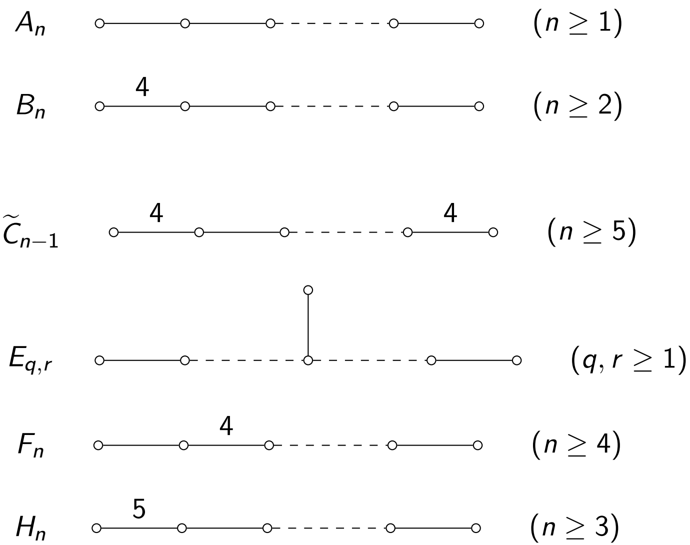
Theorem ([G–Xu])

Let (W, S) be a Coxeter system with connected Coxeter graph Γ. Then W is $a(2)$-finite if and only if either

1. Γ *is the complete graph, with any labels* ≥ 3, *or*
2. Γ *belongs to one of the six infinite families in the diagram on the following page.*

Notes

1. In each case, n is the number of vertices in the graph.
2. The dihedral types, $I_2(m)$, are included in the complete graph case.
3. The graph $E_{q,r}$ is a tree with branches of lengths 1, q and r.

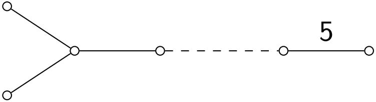

A_n $(n \geq 1)$
B_n 4 $(n \geq 2)$
$\widetilde{C}_{n-1}$ 4 4 $(n \geq 5)$
$E_{q,r}$ $(q, r \geq 1)$
F_n 4 $(n \geq 4)$
H_n 5 $(n \geq 3)$

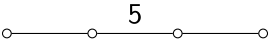
Remarks on the proof

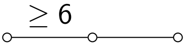
In most cases, it is relatively straightforward to prove that the diagrams listed correspond to $a(2)$-finite Coxeter groups. However, the proof that $\widetilde{C}_{n-1}$ is $a(2)$-finite for $n \geq 5$ seems to be a subtle property. It can be deduced either from Ernst's classification of the fully commutative elements, or from Shi's classification of the Kazhdan–Lusztig cells.

The bulk of the proof involves considering minimal counterexamples to $a(2)$-finiteness and proving that they are $a(2)$-infinite. In many cases, the aforementioned conjecture is known to hold, so it suffices to construct an infinite family of fully commutative elements whose heaps all have maximal antichains of size 2.

The following three cases cause particular difficulty, because (in most cases) they are neither affine Weyl groups nor star reducible Coxeter groups.







This means that in these cases, it is not straightforward to calculate the a-value of a fully commutative element from the structure of its heap.

Our approach in this case is again to construct an infinite family of fully commutative elements whose heaps have maximal antichains of size 2, and then to use Lusztig's technique of star operations to verify that all the elements in the family also have a-value 2, thus proving that the associated groups are $a(2)$-infinite.

TL; DR

- Lusztig's a-function is an algebraically defined, but somewhat mysterious, integer-valued function on a Coxeter group, defined in terms of the Kazhdan–Lusztig basis.
- An element w of a Coxeter group has a unique reduced expression if and only if $a(w) \leq 1$. A Coxeter group with a finite connected graph has finitely many elements with a-value 1 if and only if the graph has no circuits and has at most one edge with a label strictly bigger than 3.
- An element w of a Coxeter group satisfying $a(w) = 2$ must be fully commutative. A Coxeter group with a finite connected graph has finitely many elements with a-value 2 if and only if either the graph is the complete graph, or the graph belongs to one of six infinite families: A_n, B_n, $\widetilde{C}_{n-1}$, $E_{q,r}$, F_n or H_n.