Specht modules decompose as alternating sums of restrictions of Schur modules

Sami H. Assaf
Department of Mathematics
University of Southern California

David E. Speyer
Department of Mathematics
University of Michigan

27 October 2018

Spechts alternate in Schurs

Let $\mathbb{S}_\lambda(\mathbb{C}^t)$ denote Schur modules, the irred. representions of GL_t over $\mathbb{C}$ where $\ell(\lambda) \leq t$.

Let Sp_ν denote Specht modules, the irred. representations of $\mathfrak{S}_t$ over $\mathbb{C}$ where $|\nu| = t$.

For a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ and $t \geq \nu_1 + |\nu|$, set $\nu^{(t)} = (t - |\nu|, \nu_1, \nu_2, \ldots, \nu_r)$.

Since $\mathfrak{S}_t \subset \mathrm{GL}_t$, define nonnegative integers $a_\lambda^\nu(t)$ by the restriction

$$\mathrm{Res}_{\mathfrak{S}_t}^{\mathrm{GL}_t} \mathbb{S}_\lambda(\mathbb{C}^t) \cong \bigoplus_\nu \mathrm{Sp}_{\nu^{(t)}}^{\oplus a_\lambda^\nu(t)}$$

Theorem (Littlewood)

For t sufficiently large, the numbers $a_\lambda^\nu(t)$ are independent of t.

Set $a_\lambda^\nu = \lim_{t\to\infty} a_\lambda^\nu(t)$. For $|\lambda| \leq |\nu|$, we have $a_\lambda^\nu = \delta_{\lambda,\nu}$. Define integers b_λ^ν by $\left[b_\lambda^\nu\right] = \left[a_\lambda^\nu\right]^{-1}$.

In the representation ring $\mathrm{Rep}(\mathfrak{S}_t)$ for t large, we are computing change of bases:

$$[\mathbb{S}_\lambda(\mathbb{C}^t)] = \sum_\nu a_\lambda^\nu \, [\mathrm{Sp}_{\nu(t)}] \quad \Leftrightarrow \quad [\mathrm{Sp}_{\nu(t)}] = \sum_\lambda b_\lambda^\nu \, [\mathbb{S}_\lambda(\mathbb{C}^t)]$$

Theorem (Assaf–Speyer)

The integers b_λ^ν are alternating by degree. Precisely, $(-1)^{|\lambda|-|\nu|} b_\lambda^\nu \in \mathbb{N}$.

Plethystic formulas

Littlewood considered “multiplication” (*plethysmos* $\pi\lambda\eta\theta\upsilon\sigma\mu o\zeta$) of representations

$$\mathrm{GL}_m \xrightarrow{\ \psi\ } \mathrm{GL}_n \xrightarrow{\ \phi\ } \mathrm{GL}_p, \qquad \phi\circ\psi : \mathrm{GL}_m \to \mathrm{GL}_p$$

$$\left.\begin{matrix} \mathrm{char}(\psi) = g \\ \mathrm{char}(\phi) = f \end{matrix}\right\} \mathrm{char}(\phi\circ\psi) = f[g]$$

On level of characters, if $g = \sum_\alpha g_\alpha x^\alpha$ with $g_\alpha \in \mathbb{N}$, then $f[g] = f(\overbrace{x_\alpha,\ldots,x_\alpha}^{g_\alpha \text{ times}}, \overbrace{x_\beta,\ldots,x_\beta}^{g_\beta \text{ times}},\ldots)$.

Theorem (Littlewood)

Letting $s_\nu = \mathrm{ch}(\mathrm{Sp}_\nu)$ *be the* *Schur function* *and* $h_n = s_{(n)}$, *we have*

$$a^\nu_\lambda(t) = \langle s_\lambda, s_{\nu(t)}[1+h_1+h_2+h_3+\cdots]\rangle = \sum_{\mu/\nu \text{ horiz. strip}} \langle s_\lambda, s_\mu[h_1+h_2+h_3+\cdots]\rangle$$

Theorem (Assaf–Speyer)

Letting $L_m = \mathrm{ch}(\mathrm{Ind}_{C_m}^{\mathfrak{S}_m} e^{2\pi i/m}) = \frac{1}{n}\sum_{d|m}\mu(d)p_d^{m/d}$ *be the* *Lyndon symmetric function*, *we have*

$$b^\nu_\lambda = \sum_{\nu/\mu \text{ vert. strip}} (-1)^{|\nu|-|\lambda|}\langle s_{\mu^T}, s_{\lambda^T}[L_1+L_2+L_3+\cdots]\rangle$$

Tensor product multiplicities

The Kronecker coefficients $g_{\alpha,\beta,\gamma}$ give multiplicites for tensor products in $\mathfrak{S}_t$

$$\mathrm{Sp}_\alpha \otimes \mathrm{Sp}_\beta \cong \bigoplus_\gamma \mathrm{Sp}_\gamma^{\oplus g_{\alpha,\beta,\gamma}}$$

where α, β, γ are all partitions of the same size t.

Major open problem: Give a manifestly nonnegative combinatorial formula for $g_{\alpha,\beta,\gamma}$.

Murnaghan observed and Brion proved that for α, β, γ of arbitrary sizes $g_{\alpha^{(t)},\beta^{(t)},\gamma^{(t)}}$ stabilizes for t sufficiently large. The stable Kronecker coefficients $\bar{g}_{\alpha,\beta,\gamma}$ are

$$\bar{g}_{\alpha,\beta,\gamma} = \lim_{t\to\infty} g_{\alpha^{(t)},\beta^{(t)},\gamma^{(t)}}$$

Major open problem: Give a manifestly nonnegative combinatorial formula for $\bar{g}_{\alpha,\beta,\gamma}$.

The Littlewood–Richardson coefficients $c^\nu_{\lambda,\mu}$ give multiplicites for tensor products in GL_t

$$\mathbb{S}_\lambda(\mathbb{C}^t) \otimes \mathbb{S}_\mu(\mathbb{C}^t) \cong \bigoplus_\nu \mathbb{S}_\nu(\mathbb{C}^t)^{\oplus c^\nu_{\lambda,\mu}}$$

where λ, μ, ν are partitions of length at most t satisfying $|\lambda| + |\mu| = |\nu|$.

Major solved problem: Give a manifestly nonnegative combinatorial formula for $c^\nu_{\lambda,\mu}$.

Stable Specht polynomials

Using $\mathrm{char}(\mathbb{S}_\lambda(\mathbb{C}^t)) = s_\lambda(x_1, \ldots, x_t)$ allows us to solve for $c^\nu_{\lambda,\mu}$ using symmetric functions

$$s_\lambda(x_1, \ldots, x_t) s_\mu(x_1, \ldots, x_t) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, \ldots, x_t)$$

Sadly, Frobenius characters $\mathrm{ch}(\mathrm{Sp}_\nu) = s_\nu$ require Kronecker products for multiplication.

Define the (inhomogeneous) basis of stable Specht symmetric functions $s^\dagger_\nu$ by the formula

$$s_\lambda = \sum_\nu a^\nu_\lambda s^\dagger_\nu = \sum_\nu \sum_{\mu/\nu \text{ horiz. strip}} \langle s_\lambda, s_\mu[h_1 + h_2 + h_3 + \cdots] \rangle \, s^\dagger_\nu$$

Since restriction from GL_t to $\mathfrak{S}_t$ restricts with tensor product, the structure constants of the stable Specht polynomials are stable Kronecker coefficients,

$$s^\dagger_\alpha s^\dagger_\beta = \sum_\gamma \bar{g}_{\alpha,\beta,\gamma} s^\dagger_\gamma$$

Corollary (Assaf–Speyer)

The stable Specht polynomials are alternatingly Schur positive. Precisely, we have

$$s^\dagger_\nu = \sum_\lambda b^\nu_\lambda s_\lambda = \sum_\lambda \sum_{\nu/\mu \text{ vert. strip}} (-1)^{|\nu|-|\lambda|} \langle s_{\mu^T}, s_{\lambda^T}[L_1 + L_2 + L_3 + \cdots] \rangle \, s_\lambda$$

A direct combinatorial description of stable Specht polynomials might well lead to a combinatorial rule for the stable Kronecker coefficients $\bar{g}_{\alpha,\beta,\gamma}$.

Pieri's rule

Define an intermediate $\mathfrak{S}_t$-representation by $M^t_\mu = \mathrm{Ind}^{\mathfrak{S}_t}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{t-|\mu|}} \mathrm{Sp}_\mu \boxtimes \mathbb{1}_{t-|\mu|}$

The Pieri rule for induction allows us to transition between M^t_μ and $\mathrm{Sp}_{\nu(t)}$ in $\mathrm{Rep}(\mathfrak{S}_t)$.

Proposition (Assaf–Speyer)

$$[M^t_\mu] = \sum_\nu \langle s_\nu, s_\mu[1+h_1]\rangle [\mathrm{Sp}_{\nu^t}] = \sum_{\mu/\nu \text{ horiz. strip}} [\mathrm{Sp}_{\nu(t)}]$$

$$[\mathrm{Sp}_{\nu(t)}] = \sum_\mu \langle s_{\mu^T}, s_{\nu^T}[-1+h_1]\rangle [M^t_\mu] = \sum_{\nu/\mu \text{ vert. strip}} (-1)^{|\nu|-|\mu|} [M^t_\mu]$$

The representations M^t_μ arise naturally in **representations of the category of finite sets**:

- a sequence of vector spaces $V_0, V_1, V_2, \ldots$
- a functorial map $\phi_* : V_t \to V_u$ for each map $\phi : [t] \to [u]$ of finite sets

Each V_t is an $\mathfrak{S}_t$-representation, and the category of these form an abelian category.

The simple objects W_μ are indexed by partitions and $(W_\mu)_t \cong M^t_\mu$ as $\mathfrak{S}_t$-representations.

Wiltshire-Gordon showed $\mathbb{S}_\lambda(\mathbb{C}^t)$ are the projective objects, and so finding a nonnegative expansion for $[\mathbb{S}_\lambda(\mathbb{C}^t)]$ into $[M^t_\mu]$ is equivalent to finding Jordan-Holder constituents.

Conversely, expanding $[M^t_\mu]$ as an alternating sum of $[\mathbb{S}_\lambda(\mathbb{C}^t)]$ is a combinatorial shadow of the problem of finding projective resolutions of the simple objects.

Lattice of set partitions

Theorem (Assaf–Speyer)

$$[\mathbb{S}_\lambda(\mathbb{C}^t)] = \sum_\mu \langle s_\lambda, s_\mu[h_1+h_2+h_3+\cdots]\rangle [M^t_\mu]$$

Proof. We consider certain $\mathfrak{S}_l \times \mathfrak{S}_t$ representations and compare Sp_λ-isotypic components.

Let $T(l,t) = (\mathbb{C}^t)^{\otimes l}$ be the $\mathfrak{S}_l \times \mathfrak{S}_t$ rep. with basis $\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_l} \mid i_1, i_2, \ldots, i_l \in [t]\}$

By **Schur–Weyl duality**, $T(l,t) \cong \bigoplus_{|\lambda|=l} \mathrm{Sp}_\lambda \boxtimes \mathbb{S}_\lambda(\mathbb{C}^t)$, so we decompose $T(l,t)$ in another way.

Let Π_l = lattice of set partitions of $\{1,2\ldots,l\}$ ordered by refinement. For $\pi \in \Pi$ define

$$D(\pi,t) = \left\{\mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \cdots \otimes \mathbf{e}_{i_l} \in T(l,t) \mid i_p = i_q \text{ if and only if } p,q \in \pi_j \text{ for some } j\right\}$$

As an $\mathfrak{S}_l \times \mathfrak{S}_t$ rep we have $T(l,t) = \bigoplus_{|\nu|=l} D_{\mathrm{Sh}}(\nu,t)$ where $D_{\mathrm{Sh}}(\nu,t) = \bigoplus_{\mathrm{Shape}(\pi)=\nu} D(\pi,t)$.

Then $D_{\mathrm{Sh}}(\nu,t)$ is a **permutation representation**, so we can compute it by

$$D_{\mathrm{Sh}}(\nu,m) = \mathrm{Ind}^{\mathfrak{S}_l \times \mathfrak{S}_m}_{\prod \mathfrak{S}_j \wr \mathfrak{S}_{m_j}} \mathbb{1} = \sum_{\substack{|\lambda|=l \\ |\mu|=m}} \sum_{|\mu(j)|=m_j} c^\mu_{\mu(1)\mu(2)\cdots\mu(r)} \langle s_\lambda, \prod_j s_{\mu(j)}[h_j]\rangle \left[\mathrm{Sp}_\lambda \boxtimes \mathrm{Sp}_\mu\right]$$

Inducing $D_{\mathrm{Sh}}(\nu,t) = \mathrm{Ind}^{\mathfrak{S}_l \times \mathfrak{S}_t}_{\mathfrak{S}_l \times \mathfrak{S}_m \times \mathfrak{S}_{t-m}} D_{\mathrm{Sh}}(\nu,m) \boxtimes \mathbb{1}_{t-m}$ gives $M^t_\mu = \mathrm{Ind}^{\mathfrak{S}_t}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{t-|\mu|}} \mathrm{Sp}_\mu \boxtimes \mathbb{1}_{t-|\mu|}$. □

Groups acting on posets

We want to invert $T(\pi,t) = \bigoplus_{\rho\geq\pi} D(\rho,t)$ to give an expansion for $D(l,t) = D(\mathrm{Fine}_m, t)$.

Let G ($\mathfrak{S}_m$) be a **group acting on a poset** P (Π_m) with unique minimal element $\hat{0}$ (Fine_m).

Let $V = \bigoplus_{p\in P} U_p$ be a G-representation such that $g(U_p) = U_{gp}$ for each $g \in G$ and $p \in P$.

Hall showed the Möbius function $\mathfrak{m}(p)$ is the reduced Euler characteristic $\tilde{\chi}(\Delta(\hat{0},p))$.

Definition (Equivariant Möbius function (or Lefschetz element))

Let $\Delta(\hat{0},p)$ be the order complex of $(\hat{0},p)$ and let $\widetilde{H}_j$ be the reduced homology group. Then

$$\mathfrak{m}_{\mathrm{eq}}(p) = \sum_j (-1)^{j+1} \left[\widetilde{H}_j(\Delta(\hat{0},p))\right]$$

Under the map $\dim : \mathrm{Rep}(G_p) \to \mathbb{Z}$, we have $\mathfrak{m}_{\mathrm{eq}}(p)$ is sent to the Möbius function $\mathfrak{m}(p)$.

Theorem (Assaf–Speyer)

Let G_p be the stabilizer of p, and for $q \in P$, set $V_q := \bigoplus_{r\geq q} U_r$. Then in $\mathrm{Rep}(G)$ we have

$$[U_{\hat{0}}] = \sum_{p\in G\backslash P} \left[\mathrm{Ind}_{G_p}^{G} \left(\mathfrak{m}_{\mathrm{eq}}(p) \otimes V_p\right)\right]$$

The Whitehouse module

Theorem (Assaf–Speyer)

$$[M^t_\mu] = \sum_\lambda (-1)^{|\mu|-|\lambda|} \langle s_{\mu^T}, s_{\lambda^T}[L_1 + L_2 + L_3 + \cdots] \rangle [\mathbb{S}_\lambda(\mathbb{C}^t)]$$

Proof. We write the $\mathfrak{S}_m \times \mathfrak{S}_t$ rep. $D(m,t)$ in two ways and compare the Sp_μ components.

Using $D(m,t) = D(\mathrm{Fine}_m, t) = D_{\mathrm{Sh}}((1^m), t)$, from earlier we have $D(m,t) \cong \bigoplus_{|\mu|=m} \mathrm{Sp}_\mu \boxtimes M^t_\mu$.

The **equivariant Möbius inversion** formula specialized to $\mathfrak{S}_m$ acting on Π_m gives

$$[D(m,t)] = \sum_{|\nu|=m} \sum_{|\lambda|=\ell(\nu)} \left[\mathrm{Ind}^{\mathfrak{S}_m}_{G_\nu} (\mathfrak{m}_{\mathrm{eq}}(\nu) \otimes \mathrm{Sp}_\lambda) \boxtimes \mathbb{S}_\lambda(\mathbb{C}^t) \right]$$

Theorem (Sundaram and Welker)

Let Q_j be the $\mathfrak{S}_j$ rep on $\widetilde{H}_{j-3}(\Pi_j)$,

$$\widetilde{H}_{|\nu|-\ell(\nu)-2}(\Delta(\mathrm{Fine}_m, \pi)) \cong (Q_1 \S \mathbb{1}) \otimes (Q_2 \S \epsilon) \otimes (Q_3 \S \mathbb{1}) \otimes \cdots$$

Theorem (Stanley (based on Hanlon))

Q_{m-3} *vanishes for* $i \neq m-3$*, and*

$$Q_{m-3} \cong \epsilon \otimes \mathrm{Lie}_{m-3}$$

where Lie_m *is the Whitehouse module.*

The **Whitehouse module** Lie_m is the part of the free Lie algebra on $x_1, \ldots, x_m$ spanned by commutators $[\cdots[[x_{w(1)}, x_{w(2)}], x_{w(3)}], \ldots, x_{w(m)}]$ for $w \in \mathfrak{S}_m$. Brandt showed $L_m = \mathrm{ch}(\mathrm{Lie}_m)$. □

References available on the arXiv

Sami H. Assaf and David E. Speyer, *Specht modules decompose as alternating sums of restrictions of Schur modules*, arXiv:1809.10125.

Rosa Orellana and Mike Zabrocki, *Symmetric group characters as symmetric functions*, arXiv:1605.06672.

Rosa Orellana and Mike Zabrocki, *Products of characters of the symmetric group*, arXiv:1709.08098.

Thank You