Quasisymmetric Power Sums and Plethysm

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A symmetric function $f(x_1, x_2, \ldots, x_n)$ in n commuting variables is a function which remains the same when the indices of the variables are permuted.

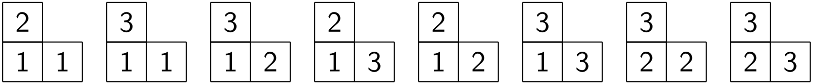
- Sym is the ring of all symmetric functions.
- Sym_n is the ring of symmetric functions in n variables.

A few different bases for Sym

- Monomial symmetric functions
  $m_{2,1}(x_1, x_2, x_3) = x_1^2 x_3 + x_1^2 x_2 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$
- Complete homogeneous symmetric functions
  $h_{2,1}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3)$
- Elementary symmetric functions
  $e_{2,1}(x_1, x_2, x_3) = (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3)$
- Power sum symmetric functions
  $p_{2,1}(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$

- <u>Schur functions</u>: $s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda)} x^T$, where $SSYT(\lambda)$ is the set of all SSYT of shape λ.

$$s_{2,1}(x_1, x_2, x_3) =$$



$$x_1^2 x_2 \; + \; x_1^2 x_3 \; + \; 2x_1x_2x_3 \; + \; x_1 x_2^2 \; + \; x_1 x_3^2 \; + \; x_2^3 x_3 \; + \; x_2 x_3^2$$

- Schur functions correspond to characters of irr reps of GL_n.
- Schur functions describe the cohomology of the Grassmannian.
- Schur functions generalize to Macdonald polynomials $(P_\lambda(X; q, t))$.

There exists a scalar product $\langle , \rangle : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathbb{C}$ defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu},$$

so that the homogeneous and monomial functions are dual. Under this pairing, we have

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu},$$

where $z_\lambda = \prod_k a_k! k^{a_k}, \quad a_k = \#\{\text{pts of length } k\}$ Ex: $z_{(3,3)} = 2!\ 3^2$.

Generating functions:

$$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i \geq 1} (1 + x_i t)$$

Note $H(t) = 1/E(-t)$.

$$P(t) = \sum_{k \geq 0} p_k t^k = \frac{d}{dt} \ln(H(t)) = \frac{d}{dt} \ln(1/E(-t))$$

The ring of noncommutative symmetric functions NSym is the $\mathbb{C}$-algebra generated freely by $\mathbf{e}_1, \mathbf{e}_2, \ldots$.

Analogous bases indexed by compositions α.

- Noncommutative elementary: $\mathbf{e}_\alpha = \mathbf{e}_{\alpha_1} \cdots \mathbf{e}_{\alpha_\ell}$. $\mathcal{A}b(\mathbf{e}_\alpha) = e_{\tilde{\alpha}}$
- Noncom. homog.: $\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_\ell}$, where $\mathbf{h}_i$ is defined by...

$$\text{if} \quad \mathbf{E}(t) = \sum_{k \geq 0} \mathbf{e}_k t^k \quad \text{and} \quad \mathbf{H}(t) = \sum_{k \geq 0} \mathbf{h}_k t^k,$$

then $\mathbf{H}(t) = 1/\mathbf{E}(-t)$. (Recall: $H(t) = 1/E(-t)$ in Sym).

$$\mathcal{A}b(\mathbf{h}_\alpha) = h_{\tilde{\alpha}}$$

⋆ Noncommutative power sums: two choices, ψ and ϕ!

	In Sym:	In NSym:
Type 1:	$P(t) = \frac{d}{dt} \ln(H(t))$	$\frac{d}{dt}\mathbf{H}(t) = \mathbf{H}(t)\mathbf{\Psi}(t)$
Type 2:	$H(t) = \exp\left(\int P(t)dt\right)$	$\mathbf{H}(t) = \exp\left(\int \mathbf{\Phi}(t)dt\right)$

Not the same! (No unique notion of log derivative for power series with noncommutative coefficients.) But

$$\mathcal{A}b(\psi_\alpha) = p_{\tilde{\alpha}} = \mathcal{A}b(\phi_\alpha)$$

A quasisymmetric function $f(x_1, x_2, \ldots, x_n)$ in n commuting variables is a function such that the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ (where $i_1 < i_2 < \cdots < i_k$) in f is equal to coefficient of $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ (where $j_1 < j_2 < \cdots j_k$) in f.

- QSym is the ring of all quasisymmetric functions.
- QSym_n is the ring of quasisymmetric functions in n variables.

A few different bases for QSym

- Monomial quasisymmetric functions

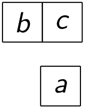
$$M_{2,1}(x_1, x_2, x_3) = x_1^2 x_3 + x_1^2 x_2 + x_2^2 x_3$$

- Gessel's fundamental quasisymmetric functions

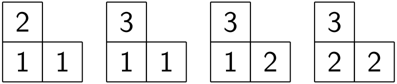
$$F_{2,1}(x_1, x_2, x_3) = x_1^2 x_3 + x_1^2 x_2 + x_2^2 x_3 + x_1 x_2 x_3$$

A filling of the Young composition diagram of shape α with numbers $1, \ldots, n$ is a semistandard Young composition tableau iff

1. The leftmost column is increasing from bottom to top and
2. the rows are increasing from left to right and
3. (YCT triple rule) for every subarray as shown, if $a \geq b$ then $a > c$, where if c is empty, $c = \infty$.



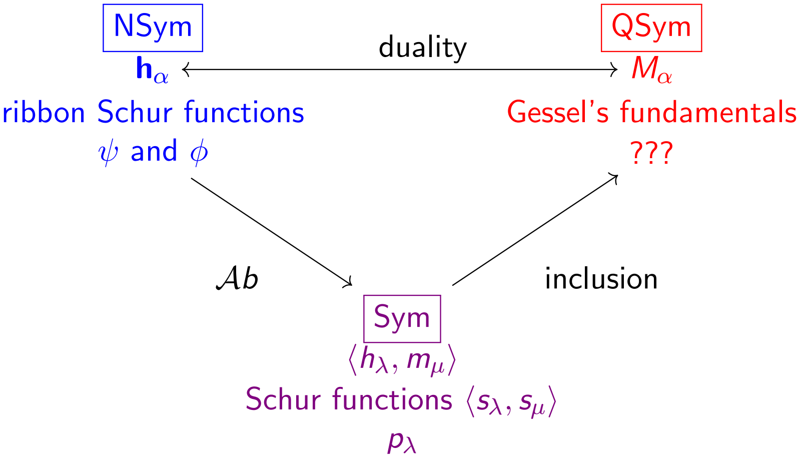
$$YQS_{2,1}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$



- $s_\lambda = \sum_{\tilde{\alpha}=\lambda} YQS_\alpha$
- If f is symmetric and Young quasisymmetric Schur-positive, the f is Schur-positive!

QSym is important!

- Relationship to posets
  (P-partitions of Gessel and Stanley)
- Terminal object in category of combinatorial Hopf algebras
  (Every combinatorial Hopf algebra maps uniquely to *QSym*)
- Dual to Solomon's descent algebra
  (Subring of group ring $\mathbb{Z}\mathfrak{S}_n$ of permutations over integers)
- Expansions into Gessel's fundamentals
  (Macdonald polynomials)
- Answer questions about symmetric functions
  (For example, Schur expansion)



Question: What is dual to ψ in QSym? to ϕ?
(Malvenuto-Reutenauer, Derksen)

Type 1

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$$

In NSym, the type 1 power sum basis ψ is defined (GKLLRT) by the generating function relation

$$\frac{d}{dt}\mathbf{H}(t) = \mathbf{H}(t)\mathbf{\Psi}(t).$$

This is equivalent to

$$\mathbf{h}_\alpha = \sum_{\beta \preccurlyeq \alpha} \frac{1}{\pi(\beta, \alpha)} \psi_\beta,$$

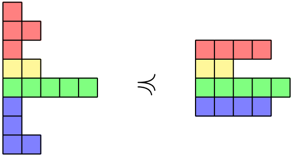
where $\pi(\beta, \alpha)$ is a combinatorial statistic on the refinement $\beta \preccurlyeq \alpha$. So, the dual in QSym will satisfy

$$\psi^*_\alpha = \sum_{\beta \succcurlyeq \alpha} \frac{1}{\pi(\alpha, \beta)} M_\beta.$$

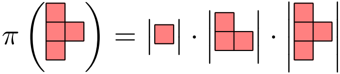
Define

$$\Psi_\alpha = z_{\tilde{\alpha}} \psi^*_\alpha, \quad \text{so that} \quad \langle \psi_\alpha, \Psi_\beta \rangle = z_{\tilde{\alpha}} \delta_{\alpha\beta}.$$

$$\boxed{\Psi_\alpha = z_{\tilde{\alpha}} \sum_{\beta \succcurlyeq \alpha} \frac{1}{\pi(\alpha,\beta)} M_\beta}$$ (diagrams courtesy of Zajj Daugherty)

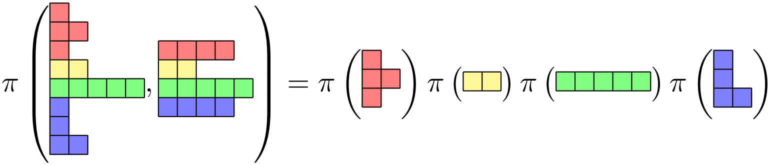


First, for each block, we compute the product of the partial sums:



$$= 1 \cdot 3 \cdot 4$$

Then, for α refining β, the coefficient of M_β in ψ^*_α is $1/\pi(\alpha,\beta)$, where



$$= (1 \cdot 3 \cdot 4)(2)(5)(1 \cdot 2 \cdot 4)$$

As another example, $z_{(2,1)} = 2$,

$$\Psi_{(1,2)} = z_{(2,1)} \psi^*_{(1,2)} = 2\left(\frac{1}{2} M_{(1,2)} + \frac{1}{3} M_{(3)}\right),$$

$$\Psi_{(2,1)} = z_{(2,1)} \psi^*_{(2,1)} = 2\left(\frac{1}{2} M_{(2,1)} + \frac{1}{6} M_{(3)}\right).$$

So

$$\Psi_{(1,2)} + \Psi_{(2,1)} = M_{(1,2)} + M_{(2,1)} + M_{(3)}$$

$$= m_{(2,1)} + m_{(3)} = m_{(2)} m_{(1)} = p_{(2)} p_{(1)} = p_{(2,1)}.$$

Theorem (BDHMN)

Type 1 QSym *power sums refine* Sym *power sums:*

$$p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha .$$

Type 2

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}.$$

In NSym, the type 2 power sum basis is defined (GKLLRT) by the generating function relation

$$\mathbf{H}(t) = \exp\left(\int \mathbf{\Phi}(t)dt\right)$$

This is equivalent to

$$\mathbf{h}_\alpha = \sum_{\beta \preccurlyeq \alpha} \frac{1}{\mathrm{sp}(\beta,\alpha)} \phi_\beta,$$

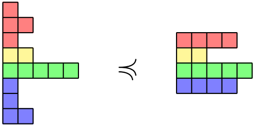
where $\mathrm{sp}(\beta,\alpha)$ is a combinatorial statistic on $\beta \preccurlyeq \alpha$.
So, the dual in QSym will satisfy

$$\phi^*_\alpha = \sum_{\beta \succcurlyeq \alpha} \frac{1}{\mathrm{sp}(\alpha,\beta)} M_\beta.$$

Define

$$\Phi_\alpha = z_{\tilde{\alpha}} \phi^*_\alpha, \qquad \text{so that} \qquad \langle \phi_\alpha, \Phi_\beta \rangle = z_\alpha \delta_{\alpha\beta}.$$

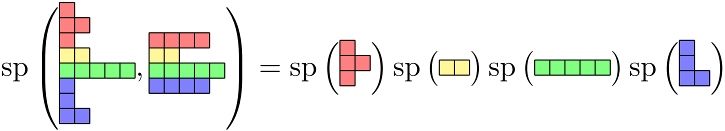
$$\boxed{\Phi_\alpha = z_{\tilde{\alpha}} \sum_{\beta \succcurlyeq \alpha} \frac{1}{\mathrm{sp}(\alpha, \beta)} M_\beta.}$$



First, for each block, we compute $\mathrm{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j$:

sp () $= 3!(1 \cdot 2 \cdot 1)$

Then, for α refining β, the coeff of M_β in ψ^*_α is $1/\mathrm{sp}(\alpha, \beta)$, where



$$= 3!(1 \cdot 2 \cdot 1) \cdot 1!(2) \cdot 1!(5) \cdot 3!(1 \cdot 1 \cdot 2)$$

As another example, $z_{(2,1)} = 2$,

$$\Phi_{(1,2)} = z_{(2,1)} \phi^*_{(1,2)} = 2\left(\frac{1}{2} M_{(1,2)} + \frac{1}{4} M_{(3)}\right),$$

$$\Phi_{(2,1)} = z_{(2,1)} \phi^*_{(2,1)} = 2\left(\frac{1}{2} M_{(2,1)} + \frac{1}{4} M_{(3)}\right).$$

So

$$\Phi_{(1,2)} + \Phi_{(2,1)} = M_{(1,2)} + M_{(2,1)} + M_{(3)}$$

$$= m_{(2,1)} + m_{(3)} = m_{(2)} m_{(1)} = p_{(2)} p_{(1)} = p_{(2,1)}.$$

Theorem (BDHMN)

Type 2 QSym *power sums refine* Sym *power sums:*

$$p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Phi_\alpha.$$

Results about quasisymmetric power sums

- $\Psi_\alpha \Psi_\beta = \frac{z_\alpha z_\beta}{z_{\alpha\cdot\beta}} \sum_{\gamma \in \alpha ⧢ \beta} \Psi_{\mathrm{wd}(\gamma)}$ (BDHMN)
- $\Phi_\alpha \Phi_\beta = \frac{z_\alpha z_\beta}{z_{\alpha\cdot\beta}} \sum_{\gamma \in \alpha ⧢ \beta} \Phi_{\mathrm{wd}(\gamma)}$ (BDHMN)
- The set of quasisymmetric power sums indexed by Lyndon words forms a multiplicative basis for QSym.
- Formulas for expansion into fundamentals (BDHMN)
- Formulas for the expansion of fundamentals into quasisymmetric power sums (Alexandersson-Sulzgruber)
- Expansions of generating functions of reverse P-partitions into quasisymmetric power sums (Alexandersson-Sulzgruber)

Plethysm

- V, W, Y finite dimensional complex vector spaces
- $GL(V)$ = group of invertible linear transformations $A : V \to V$ (under composition)
- polynomial reps $h : GL(V) \to GL(W), \quad k : GL(W) \to GL(Y)$
- composition $kh : GL(V) \to GL(W)$ defines a polynomial representation of $GL(V)$

Then $char(h) = \sum_{i=1}^{N} \theta^{a^i}$, where the θ^a are monomials in the eigenvalues of some $A \in GL(V)$.

$$\boxed{char(kh) = char(k)(\theta^{a^1}, \ldots, \theta^{a^N})}$$

Given a function $f = \sum_{i \geq 1} x^{a^i} \in \text{Sym}$ and $g \in \text{Sym}$, the plethysm $g[f]$ is defined by

$$g[f] = g(x^{a^1}, x^{a^2}, \ldots).$$

Kostka-Foulkes Conjecture:

$$\text{Schur positivity of } s_m[s_n] - s_n[s_m]$$

On the symmetric power sums, plethysm is beautiful:

$$p_m[p_n] = p_{mn}, \quad p_{a,b}[p_{c,d}] = p_{ac,ad,bc,bd}, \text{ etc...}$$

Natural (naive) question:

Is there a "nice" formula for $\psi_\alpha[\psi_\beta]$? (similarly for $\phi_\alpha[\phi_\beta]$?)

Kostka-Foulkes Conjecture:

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Natural (naive) question:

Is there a “nice” formula for $\psi_\alpha[\psi_\beta]$? (similarly for $\phi_\alpha[\phi_\beta]$?)

The short answer is “no”!

A slightly longer answer

For plethysm on QSym, the order of the variables matters:

$$M_{21}[\Psi_2(x_1, x_2)] = M_{21}[x_1^2 + x_2^2] = M_{21}(x_1^2, x_2^2) = x_1^4 x_2^2 = M_{42}$$

$$M_{21}[x_2^2 + x_1^2] = M_{21}(x_2^2, x_1^2) = x_2^4 x_1^2 = M_{24}$$

Is there a natural order on monomials under which a “nice” formula for $\psi_\alpha[\psi_\beta]$ exists?

$$\begin{aligned} p_{21}[p_{11}] &= p_{2211} \\ &= \psi_{2211} + \psi_{2121} + \psi_{2112} + \psi_{1122} + \psi_{1212} + \psi_{1221} \\ &= (\psi_{21} + \psi_{12})[\psi_{11}] \\ &= \psi_{21}[\psi_{11}] + \psi_{12}[\psi_{11}] \end{aligned}$$

Still “no”, but for some positivity...

Theorem (Loehr-Remmel, 2011)

Let $\mathcal{A}$ be an ordered alphabet consisting of signed monomials. The signed weight $wt(\mathcal{A})$ of the alphabet is obtained by summing the signed monomials in order. Then

$$s_\lambda[wt(\mathcal{A})] = \sum_{T \in SSYT_{\mathcal{A}}(\lambda)} wt(T).$$

Corollary (Allen, M, Moore, 2018)

Similarly,

$$YQS_\lambda[wt(\mathcal{A})] = \sum_{T \in YCT_{\mathcal{A}}(\lambda)} wt(T).$$

Useful since a symmetric function with a positive quasisymmetric Schur expansion is automatically Schur positive.

Several References

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THANK YOU!