$sl(\infty)$-modules arising from categorical action on the category $\mathcal{O}$ for general linear superalgebra.

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Classical Lie superalgebra $\mathfrak{gl}(m|n)$

Let $V = V_0 \oplus V_1$ be a vector superspace of dimension $(m|n)$. The general linear Lie superalgebra $\mathfrak{g} := \mathfrak{gl}(m|n)$ is the algebra $\mathrm{End}_k(V)$ of linear transformations of V

- the supercommutator $[X, Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$ sign rule;
- Elements are matrices $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$,
- even part $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$, odd part $\left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right)$.

Invariant symmetric form: $\mathrm{str}\, XY$ leads to the invariant element in $\mathfrak{g} \otimes \mathfrak{g}$:

$$\Omega := \sum (-1)^{\bar{X}_i} X_i \otimes X^i.$$

If M, N are $\mathfrak{gl}(m|n)$-modules, then $\Omega : M \otimes N \to M \otimes N$ commutes with $\mathfrak{sl}(\infty)$-action.

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Category $\mathcal{O}$

- $\mathfrak{b}$ subalgebra of upper triangular matrices, $\mathfrak{h}$ Cartan subalgebra;
- $\mathcal{O}_{m|n}$ the category of modules, semisimple over $\mathfrak{h}$ with integral weights, locally finite over $\mathfrak{b}$ and finitely generated;
- Highest weight category. Standard objects are Verma modules $M(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} C_\lambda$;
- Kazhdan–Lusztig theory: Cheng–Lam–Wang, Brundan–Losev–Webster.

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Brundan's categorification

- Let $i \in \mathbb{Z}$, $M \in \mathcal{O}_{m|n}$, let $E_i M$ (resp. $F_i M$) be the generalized eigenspace of Ω in $M \otimes V$ (resp. $M \otimes V^*$) with eigenvalue i. Mutually adjoint exact functors E_i and $F_i : \mathcal{O}_{m|n} :\to \mathcal{O}_{m|n}$.
- Let $\mathsf{K}_{m|n}$ be the complexified reduced Grothendieck group of $\mathcal{O}_{m|n}$. Then E_i, F_i induce linear operators $e_i, f_i : \mathsf{K}_{m|n} \to \mathsf{K}_{m|n}$.
- $e_i, f_i, i \in \mathbb{Z}$ satisfy Serre's relation for the infinite-dimensional Lie algebra $\mathfrak{sl}(\infty)$ with the Dynkin diagram A_∞

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Goals

- Describe the structure of $\mathsf{K}_{m|n}$ as an $\mathfrak{sl}(\infty)$-module. Compute the socle filtration of $\mathsf{K}_{m|n}$ and understand its categorical meaning.
- Relate $\mathfrak{sl}(\infty)$-morphisms $\mathsf{K}_{m|n} \to \mathsf{K}_{m-1|n-1}$ with certain tensor functors $DS : \mathcal{O}_{m|n} \to \mathcal{O}_{m-1|n-1}$.
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Example.

$\mathcal{O}_{1|1}$ coincides with the category $\mathcal{F}_{1|1}$ of finite-dimensional $\mathfrak{gl}(1|1)$-modules. Let us describe the $\mathfrak{sl}(\infty)$-module $\mathbf{K}_{1|1}$. Let $\mathbf{E}$ and $\mathbf{E}_*$ denote the natural and conatural $\mathfrak{sl}(\infty)$-modules. We have the non-split exact sequence

$$0 \to \mathbf{P}_{1|1} \to \mathbf{E} \otimes \mathbf{E}_* \xrightarrow{\text{tr}} \mathbb{C} \to 0,$$

where $\mathbf{P}_{1|1}$ corresponds to the Grothendieck subgroup of $\mathbf{K}_{1|1}$ generated by classes of projective modules. Furthermore, $\mathbf{E} \otimes \mathbf{E}_*$ is identified with the Grothendieck subgroup generated by classes of Verma modules. However, $\mathbf{K}_{1|1} \neq \mathbf{E} \otimes \mathbf{E}_*$. We have another non-split exact sequence

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To describe the $\mathfrak{sl}(\infty)$-structure of $\mathbf{K}_{1|1}$ identify $\mathbf{E} \otimes \mathbf{E}_*$ with the Lie algebra $\mathfrak{gl}(\infty)$ of finitary infinite matrices $\{(a_{ij})\}_{i,j\in\mathbb{Z}}$. Then $\mathbf{K}_{1|1} = \mathfrak{gl}(\infty) \oplus \mathbb{C}J$, where $J_{ij} = \operatorname{sgn}(i)\delta_{ij}$.

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Continuation of the example. Socle filtration.

$$\mathsf{P}_{1|1} = \operatorname{soc} \mathsf{K}_{1|1} \simeq \mathfrak{sl}(\infty), \quad \mathsf{K}_{1|1}/\mathfrak{sl}(\infty) = \mathbb{C}^2.$$

$$\dim \operatorname{Hom}_{\mathfrak{sl}(\infty)}(\mathsf{K}_{1|1}, \mathbb{C}) = 2.$$

Choose a basis in $\mathfrak{g}_1$:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Define the functors $\mathcal{O}_{1|1} \to \mathsf{Svect}$ by

$$DS_x M = \operatorname{Ker} x_M / \operatorname{Im} x_M, \quad DS_y M = \operatorname{Ker} y_M / \operatorname{Im} y_M.$$

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A category of $\mathfrak{sl}(\infty)$-modules.

Definition

Let $\mathcal{T}$ denote the full subcategory of $\mathfrak{sl}(\infty)$-modules consisting of modules U satisfying the following conditions:

1. U is an integrable module of finite length.
2. Simple constituents of U are tensor modules $S(\lambda, \mu) \subset \mathbf{E}^{\otimes|\lambda|} \otimes \mathbf{E}_*^{\otimes|\mu|}$ (here (λ, μ) is a bipartition).
3. For any $u \in U$ we have $e_i u = f_i u = 0$ for all but finitely many $i \in \mathbb{Z}$.

Theorem

The category $\mathcal{T}$ has enough injective objects. The injective hull $I(\lambda, \mu)$ of $S(\lambda, \mu)$ has the socle filtration:

$$[\frac{\mathrm{soc}_{k+1} I(\lambda,\mu)}{\mathrm{soc}_k I(\lambda,\mu)} : S(\lambda',\mu')] = \sum_{|\gamma|+|\delta|=k} N^{\lambda}_{\lambda',\gamma,\delta} N^{\mu}_{\mu',\gamma,\delta}$$

where $N^{\nu}_{\nu',\gamma,\delta}$ are the Littlewood–Richardson coefficients.

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Main results.

Theorem

1. $\mathsf{K}_{m|n}$ *is an injective object of* $\mathcal{T}$.
2. *The submodule* $\mathbf{E}^{\otimes m} \otimes \mathbf{E}_*^{\otimes n} \hookrightarrow \mathsf{K}_{m|n}$ *is isomorphic to the subgroup generated by the classes of all Verma modules.*
3. *The socle of* $\mathsf{K}_{m|n}$ *is isomorphic to the subgroup generated by the classes of all projective objects in* $\mathcal{O}_{m|n}$.

The Zuckerman functor

Let $\mathcal{F}_{m|n}$ be the category of finite-dimensional $\mathfrak{g} := \mathfrak{gl}(m|n)$-modules semisimple over $\mathfrak{h}$ and $\mathbf{J}_{m|n}$ denote its complexified reduced Grothendieck group.

For $M \in \mathcal{O}_{m|n}$ denote by ΓM the subset of all $\mathfrak{g}_0$-finite vectors. Then Γ defines a left exact functor $\mathcal{O}_{m|n} \to \mathcal{F}_{m|n}$. Its derived functor Γ^i is called the Zuckerman functor.

Theorem

1. $\mathbf{J}_{m,n}$ *is the injective hull of* $S(1^m, 1^n)$
2. *The map* $[M] \to \sum(-1)^i[\Gamma^i M]$ *defines an* $\mathfrak{sl}(\infty)$*-equivariant map* $\gamma : \mathbf{K}_{m|n} \to \mathbf{J}_{m|n}$.
3. *The restriction of* γ *to* $\mathbf{E}^{\otimes m} \otimes \mathbf{E}_*^{\otimes n} \hookrightarrow \mathbf{K}_{m|n}$ *coinsides with the natural projector to* $\Lambda^m \mathbf{E} \otimes \Lambda^n \mathbf{E}_*$.

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Then Γ defines a left exact functor $\mathcal{O}_{m|n} \to \mathcal{F}_{m|n}$. Its derived functor Γ^i is called the Zuckerman functor.

Theorem

1. $\mathbf{J}_{m,n}$ *is the injective hull of* $S(1^m, 1^n)$
2. *The map* $[M] \to \sum(-1)^i[\Gamma^i M]$ *defines an* $\mathfrak{sl}(\infty)$*-equivariant map* $\gamma : \mathbf{K}_{m|n} \to \mathbf{J}_{m|n}$.
3. *The restriction of* γ *to* $\mathbf{E}^{\otimes m} \otimes \mathbf{E}_*^{\otimes n} \hookrightarrow \mathbf{K}_{m|n}$ *coinsides with the natural projector to* $\Lambda^m \mathbf{E} \otimes \Lambda^n \mathbf{E}_*$.

DS functors.

For $\mathfrak{g} = \mathfrak{gl}(m|n)$ let

$$X := \{x \in \mathfrak{g}_1 \mid [x,x] = 0\},$$

$$X_k := \{x \in \mathfrak{g}_1 \mid [x,x] = 0, \mathrm{rk}(x) = k\}, \;\; k \leq \min(m,n).$$

For a $\mathfrak{g}$-module M define $DS_x M := \mathbf{ker} x_M / \mathbf{im} x_M$.

Theorem

Let $x \in X_k$.

1. $DS_x : \mathcal{O}_{m|n} \to \mathcal{O}_{m-k|n-k}$ *is a symmetric monoidal functor which commutes with translation functors* E_i, F_i.
2. *Passage to the Grothendieck groups induces a homomorphism of* $\mathfrak{sl}(\infty)$*-modules* $ds_x : \mathbf{K}_{m|n} \to \mathbf{K}_{m-k|n-k}$.

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Remark.

Although DS_x is not exact, for an exact sequence $0 \to N \to M \to L \to 0$ we have a canonical exact sequence

$$0 \to R \to DS_x N \to DS_x M \to DS_x L \to R \otimes \mathbb{C}^{0|1} \to 0.$$

This insures the existence of the corresponding map ds_x for the reduced Grothendieck groups.

Conjecture. Let $\mathsf{K}^k_{m|n}$ be the subgroup in $\mathsf{K}_{m|n}$ generated by the classes of all modules M such that $DS_x M = 0$. Then $\mathrm{soc}_{k+1}\,\mathsf{K}_{m|n} = \mathsf{K}^k_{m|n}$.

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