A combinatorial description of some representations of degenerate affine Hecke algebras of type BC

Yue Zhao

University of California, Davis

October 27, 2018

Outline

Irreducible $\mathcal{Y}$-semisimple representations of degenerate AHA of type BC

Some representations defined by the Etingof-Freund-Ma functor

$$\mathcal{F} : \{G - modules\} \to \{\dot{H}_n - representations\}$$

Generators and Relations of the dAHA of type BC_n

The degenerate affine Hecke algebra $\dot{H}_n(\kappa_1, \kappa_2)$ over the field $\mathbb{C}$ is an algebra with generators

$$s_1, \ldots, s_{n-1}, \gamma, y_1, y_2, \cdots, y_n;$$

and relations

- quadratic relations $s_i^2 = 1$, $\gamma^2 = 1$,
- braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$,
  $s_i s_j = s_j s_i$,
  $s_{n-1}\gamma s_{n-1}\gamma = \gamma s_{n-1}\gamma s_{n-1}$,
  $s_i\gamma = \gamma s_i$;
- $y_i y_j = y_j y_i$,
- $s_i y_i - y_{i+1} s_i = \kappa_1$,
- $\gamma y_n + y_n \gamma = \kappa_2$.

Let $W = \langle s_1, \ldots, s_{n-1}, \gamma\rangle / \sim$ be the Weyl group of type BC.

$\mathcal{Y}$-semisimplicity

Since the subalgebra $\mathcal{Y} = \mathbb{C}\{Y_1, \dots, Y_n\}$ is commutative, then we can consider $\mathcal{Y}$-eigenspaces. Let M be an $\dot{H}_n$ module, for each function $\zeta : \{1, 2, \dots, n\} \to \mathbb{C}$, define the subspace of M

$$M_\zeta = \{v \in M | y_i v = \zeta(i) v \text{ for } i \in [1, n]\}.$$

If

$$M = \bigoplus M_\zeta,$$

in this case we call M $\mathcal{Y}$-semisimple.
Each ζ such that $M_\zeta \neq 0$ is called a $\mathcal{Y}$-weight. M_ζ is the $\mathcal{Y}$-weight space of $\mathcal{Y}$-weight ζ and $v \neq 0 \in M_\zeta$ is a weight vector of weight ζ.

Intertwining operators

For each $i \in [0, n-1]$, define

$$\phi_i = [s_i, y_i],$$

and for γ, define

$$\phi_n = [\gamma, y_n].$$

For each $w \in W$, it has a reduced expression $w = s_{i_1} s_{i_2} \ldots s_{i_m}$. Define

$$\phi_w = \phi_{i_1} \phi_{i_2} \ldots \phi_{i_m}.$$

This is well-defined since $\phi_1, \ldots, \phi_n$ satisfy the same braid relations as $s_1, \ldots, s_{n-1}, \gamma$.

Facts about intertwining operators

$$y_i \phi_i = \phi_i y_{i+1},$$

$$y_n \phi_n = -\phi_n y_n.$$

So

$$\phi_w M_\zeta \subset M_{w \cdot \zeta}.$$

$$\begin{array}{ccccc} v & \xrightarrow{\phi_2} & \phi_2 v & \xrightarrow{\phi_1} & \phi_1 \phi_2 v \\ [-1,-2] & \xrightarrow{\gamma} & [-1,2] & \xrightarrow{s_1} & [2,-1] \end{array}$$

Facts about irreducible $\mathcal{Y}$-semisimple representations

Let M be an irreducible and $\mathcal{Y}$-semisimple representation of $\dot{H}_n$.

- Fix a $\mathcal{Y}$-weight vector v, the set $\{\phi_w v | w \in W\}$ spans M.
- Every $\mathcal{Y}$-weight space of M is 1-dimensional.
- Let ζ be a weight of M such that $\zeta(i) - \zeta(i+1) = \pm 1$ for some $i \in [0, n-1]$. Then $\phi_i v = 0$ for $v \in M_\zeta$. Let ζ be a weight of M such that $\zeta(n) = \pm \frac{\kappa_2}{2}$. Then $\phi_n v = 0$ for $v \in M_\zeta$.

Example: $\kappa_2 = -2$

$$[-1,-2] \overset{\gamma}{\text{———}} [-1,2] \overset{s_1}{\text{———}} [2,-1]$$

Etingof-Freund-Ma functor

Set $N = p + q$. Let $G = GL_N$ and $K = GL_p \times GL_q$. Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_n$ of G and $\mathfrak{k}$ be the Lie algebra of K. Let $\mathfrak{k}_0$ be the subalgebra of $\mathfrak{k}$ consisting of all the trace 0 matrices.

$$\mathcal{F}_{n,\mu} : \{G - modules\} \to \{\dot{H}_n(1, p - q - \mu N) - representations\}$$

Let V be the defining representation, define $\mathcal{F}_{n,\mu}(M) = (M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$, which is the μ-invariant of the tensor, i.e. $xv = \mu\chi(x)v$ for every $x \in \mathfrak{k}_0$, where $\mu \in \mathbb{C}$ and χ is a character of $\mathfrak{k}$ which is

$$\chi(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}) = qtrX_1 - ptrX_2.$$

Define the actions of s_i, γ and y_k

For a tensor $m \otimes v^{(1)} \cdots v^{(n)}$ in $\mathcal{F}_{n,\mu}(M)$, where $v^{(i)}$ denotes the i-th copy, s_i flips $v^{(i)}$ and $v^{(i+1)}$ and γ acts on $v^{(n)}$ by

$$J = \begin{bmatrix} I_p & 0 \\ o & -I_q \end{bmatrix}$$

$$y_k = -\Sigma_{i|j} E_{ij}^{(0)} \otimes E_{ji}^{(k)} + \frac{1}{2}\Sigma_{j>k} S_{kj} - \frac{1}{2}\Sigma_{j<k} S_{kj} + \frac{1}{2}\Sigma_{j\neq k} S_{kj}\gamma_k\gamma_j + \frac{\kappa_2}{2}\gamma_k,$$

where E_{ij}, $S_{ij} = (ij)$ and $\gamma_k = S_{kn}\gamma S_{kn}$.

Example of $G = GL_2$, $p = q = 1$

Let M be a finite dimensional irreducible representation of G. The irreducible representations of GL_2 are indexed by young diagrams with two rows. So the irreducible representation M with the highest weight $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is denoted by the young diagram

$$V_{\square\square}$$

In this case, $\mathfrak{t}_0$ is only one dimensional with the basis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and we have the character

$$\chi(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = 2.$$

So $xv = \mu\chi(x)v$ for every $x \in \mathfrak{t}_0$ means $\mathcal{F}_{n,\mu}(V_{\square\square})$ is spanned by vectors of degree 2μ.

By Pieri rule, the tensor product decomposes to the sum of finite dimensional irreducible representations of GL_2. They are indexed by standard skew young tableaux with number 1 and 2.

$$V_{\ydiagram{2}} \otimes V^{\otimes 2}$$

$$= V_{\ytableaushort{{}{}12}} \oplus V_{\ytableaushort{{}{}1,2}} \oplus V_{\ytableaushort{{}{}2,1}} \oplus V_{\ytableaushort{{}{},12}}$$

The weight space decomposition of each irreducible is as follows.

$$V_{\begin{array}{|c|c|c|c|}\hline \ & \ & 1 & 2 \\ \hline \end{array}} = \begin{pmatrix}4\\0\end{pmatrix} \oplus \begin{pmatrix}3\\1\end{pmatrix} \oplus \begin{pmatrix}2\\2\end{pmatrix} \oplus \begin{pmatrix}1\\3\end{pmatrix} \oplus \begin{pmatrix}0\\4\end{pmatrix}$$

$$V_{\begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline 2 \\ \cline{1-1} \end{array}} = \begin{pmatrix}3\\1\end{pmatrix} \oplus \begin{pmatrix}2\\2\end{pmatrix} \oplus \begin{pmatrix}1\\3\end{pmatrix}$$

$$V_{\begin{array}{|c|c|c|}\hline \ & \ & 2 \\ \hline 1 \\ \cline{1-1} \end{array}} = \begin{pmatrix}3\\1\end{pmatrix} \oplus \begin{pmatrix}2\\2\end{pmatrix} \oplus \begin{pmatrix}1\\3\end{pmatrix}$$

$$V_{\begin{array}{|c|c|}\hline \ & \ \\ \hline 1 & 2 \\ \hline \end{array}} = \begin{pmatrix}2\\2\end{pmatrix}$$

The set of all the blue pieces spans $\mathcal{F}_{2,0}(V_{\Box\Box})$. The blue piece in $V_{\Box\Box\boxed{1}\boxed{2}}$ is the following vector.

$$\color{blue}{\binom{2}{2}}\color{black} = 2\binom{0}{2}\otimes v_+\otimes v_+ + 2\binom{1}{1}\otimes v_-\otimes v_+ + 2\binom{1}{1}\otimes v_+\otimes v_- + 2\binom{2}{0}\otimes v_-\otimes v_-$$

This is $\mathcal{Y}$-weight vector of the weight $[-1, -2]$.

Start with this $\mathcal{Y}$-weight vector we will have the other three $\mathcal{Y}$-weight vectors.

$$[-1,-2] \overset{\gamma}{\overline{\qquad\qquad}} [-1,2] \overset{s_1}{\overline{\qquad\qquad}} [2,-1] \overset{\gamma}{\overline{\qquad\qquad}} [2,1]$$

Since the space $\mathcal{F}_{2,0}(V_{\square\square})$ is of dimension 4, so it is a $\mathcal{Y}$-semisimple representation.

The set of all the red pieces spans $\mathcal{F}_{2,1}(V_{\square\square})$. The red piece in $V_{\square\square\boxed{1}\boxed{2}}$ is the following vector.

$$\color{red}{\binom{3}{1}} = \binom{1}{1} \otimes v_+ \otimes v_+ + \binom{2}{0} \otimes v_- \otimes v_+ + \binom{2}{0} \otimes v_+ \otimes v_-$$

This is $\mathcal{Y}$-weight vector of the weight $[-1,-2]$.

$$[-1,-2] \overset{\gamma}{\text{———}} [-1,2] \overset{s_1}{\text{———}} [2,-1]$$